The trigonometric BC_n Sutherland system: action-angle duality and applications

Joint work with László Fehér

Tamás F. Görbe

Department of Theoretical Physics
University of Szeged

Budapest University of Technology and Economics
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This talk is based on the papers:

L. Fehér and T.F. Görbe
Duality between the trigonometric BC_n Sutherland system and a completed rational Ruijsenaars – Schneider – van Diejen system
J.Math.Phys. **55** (2014) 102704, **arXiv:1407.2057 [math-ph]**

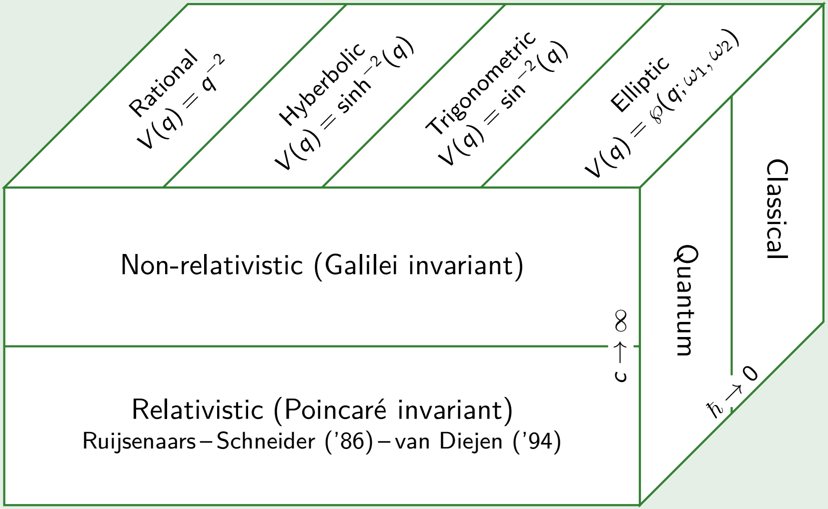
T.F. Görbe
On the derivation of Darboux form for the action-angle dual of trigonometric BC_n Sutherland system
J. Phys.: Conf. Ser. **563** (2014) 012012, **arXiv:1410.0301 [math-ph]**

T.F. Görbe and L. Fehér
Equivalence of two sets of Hamiltonians associated with the rational BC_n Ruijsenaars-Schneider-van Diejen system
Phys. Lett. A **379**, 2685-2689 (2015), **arXiv:1503.01303 [math-ph]**

Outline

Calogero ('71) – Moser ('75) – Sutherland ('71-72)

Integrable many-body systems describing particles moving on a line or circle. Connected to many branches of mathematics and physics.



Generalizations associated with root systems, Olshanetsky – Perelomov ('76)

Action-angle duality in general

Two Liouville integrable Hamiltonian systems (M, ω, H) and $(\tilde{M}, \tilde{\omega}, \tilde{H})$ with Darboux coordinates q, p and λ, θ, resp. are **action-angle duals** of each other if there is a **global symplectomorphism** $\mathcal{R}\colon M \to \tilde{M}$ such that $(\lambda, \theta) \circ \mathcal{R}$ are action-angle variables for H and $(q, p) \circ \mathcal{R}^{-1}$ are action-angle variables for $\tilde{H}$.

Duality relations of CMS and RSvD systems, Ruijsenaars ('88-'95)

$$
\begin{array}{rcl}
\text{rat. CMS} & \xleftrightarrow{\ \mathcal{R}\ } & \text{rat. CMS} \\
r \to \infty \uparrow & & \uparrow c \to \infty \\
\text{hyp. CMS} & \xleftrightarrow{\ \mathcal{R}\ } & \text{rat. RSvD} \\
c \to \infty \uparrow & & \uparrow r \to \infty \\
\text{hyp. RSvD} & \xleftrightarrow{\ \mathcal{R}\ } & \text{hyp. RSvD}
\end{array}
$$

Historically, the first example was the self-dual rational Calogero-Moser system, its interpretation in terms of symplectic reduction is due to Kazhdan, Kostant and Sternberg ('78).

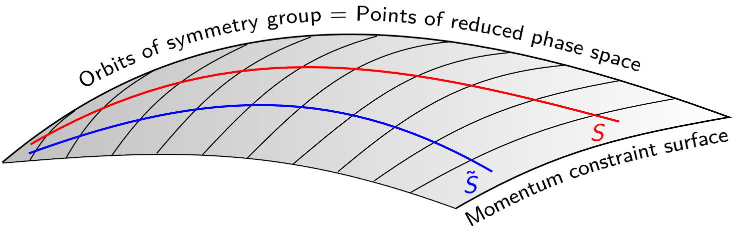
The basic idea, Gorsky et al. ('94-2000)

Start with "big phase space", of group theoretic origin, equipped with two canonical families of commuting "free" Hamiltonians.

Apply suitable single (symplectic) reduction to the big phase space and construct two "natural" models (S and $\tilde{S}$) of the reduced phase space.

The two families of "free" Hamiltonians turn into interesting many-body Hamiltonians and particle-position variables in terms of both models. Their rôle is interchanged in the two models.

The natural symplectomorphism between the two models of the reduced phase space yields the duality symplectomorphism.



Our aim was to construct the dual model of the trigonometric BC_n Sutherland system by implementing the reduction-based technique mentioned above.

Hamiltonian of the trigonometric BC_n Sutherland system

$$H(q,p)=\frac{1}{2}\sum_{j=1}^{n}p_j^2+\sum_{1\leq j<k\leq n}\frac{\gamma}{\sin^2(q_j\pm q_k)}+\sum_{j=1}^{n}\frac{\gamma_1}{\sin^2(q_j)}+\sum_{j=1}^{n}\frac{\gamma_2}{\sin^2(2q_j)},$$

$\boldsymbol{\gamma},\boldsymbol{\gamma_1},\boldsymbol{\gamma_2}$ are **real coupling constants** satisfying $\gamma>0$, $\gamma_2>0$, $4\gamma_1+\gamma_2>0$.
Hamiltonian of dual RSvD-type system

$$\tilde{H}^0(\lambda,\vartheta)=\sum_{j=1}^{n}\cos(\vartheta_j)\left[1-\frac{\nu^2}{\lambda_j^2}\right]^{\frac{1}{2}}\left[1-\frac{\kappa^2}{\lambda_j^2}\right]^{\frac{1}{2}}\prod_{\substack{k=1\\(k\neq j)}}^{n}\left[1-\frac{4\mu^2}{(\lambda_j\pm\lambda_k)^2}\right]^{\frac{1}{2}}$$

$$+\frac{\nu\kappa}{4\mu^2}\prod_{j=1}^{n}\left[1-\frac{4\mu^2}{\lambda_j^2}\right]-\frac{\nu\kappa}{4\mu^2},$$

with $\boldsymbol{\mu},\boldsymbol{\nu},\boldsymbol{\kappa}$ **real coupling constants** satisfying $\mu>0$, $\nu>|\kappa|\geq 0$.
Duality will be established under the following relation between the coupling parameters

$$\gamma=\mu^2,\quad \gamma_1=\frac{\nu\kappa}{2},\quad \gamma_2=\frac{(\nu-\kappa)^2}{2}.$$

- Unitary group of degree $2n$ | $G := \mathrm{U}(2n) = \{y \in \mathrm{GL}(2n,\mathbb{C}) \mid y^\dagger y = \mathbf{1}_{2n}\}$
- Involutive automorphism | $\sigma\colon G \to G,\ y \mapsto \sigma(y) := CyC^{-1},\ C := \begin{bmatrix} \mathbf{0}_n & \mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{bmatrix}$
- Important subsets | $G_\pm := \{\sigma(y) = y^{\pm 1}\}$ | Symmetry group | $G_+ \times G_+$
- Lie algebra decomposition | $\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-$, with $\mathcal{G}_\pm := \ker(\sigma \mp \mathrm{id})$
- Coadjoint orbit | $V_\mathbb{R} = [1\ \ldots\ 1\ -1\ \ldots\ -1]^\top$

$$\upsilon^\ell_{\mu,\nu}(V) := \mathrm{i}\mu\big(VV^\dagger - \mathbf{1}_{2n}\big) + \mathrm{i}(\mu-\nu)C \in \mathcal{G}_+ \quad (\mu,\nu \in \mathbb{R})$$

The coadjoint orbit passing through $\upsilon^\ell_\mathbb{R} := \upsilon^\ell_{\mu,\nu}(V_\mathbb{R})$

$$\mathcal{O}^\ell := \big\{\upsilon^\ell_{\mu,\nu}(V) \mid V \in \mathbb{C}^{2n},\ |V|^2 = 2n,\ CV + V = 0\big\}$$

One point coadjoint orbit $\mathcal{O}^r \equiv \{\upsilon^r\} \equiv \{-\mathrm{i}\kappa C\} \quad (\kappa \in \mathbb{R})$, $\mathcal{O} := \mathcal{O}^\ell \oplus \mathcal{O}^r$

- Extended phase space | $P := T^*G \times \mathcal{O}$, $\Omega = \Omega^{T^*G} + \Omega^{\mathcal{O}}$
- Action of symmetry group | $g_L, g_R \in G_+$, $(y, Y, \upsilon^\ell \oplus \upsilon^r) \in P$

$$\Phi_{(g_L,g_R)}(y, Y, \upsilon^\ell \oplus \upsilon^r) = \big(g_L y g_R^{-1}, g_R Y g_R^{-1}, g_L \upsilon^\ell g_L^{-1} \oplus \upsilon^r\big)$$

- Momentum map | $J(y, Y, \upsilon^\ell \oplus \upsilon^r) = \big((yYy^{-1})_+ + \upsilon^\ell\big) \oplus \big(-Y_+ + \upsilon^r\big)$
- Reduced phase space | $P_{\mathrm{red}} = J^{-1}(0)/(G_+ \times G_+) =: P_0/(G_+ \times G_+)$

- Maximal Abelian subalgebra in $\mathcal{G}_-$ | $\mathcal{A} := \{\mathrm{i}Q(q) = \mathrm{i}\,\mathrm{diag}(q, -q) \mid q \in \mathbb{R}^n\}$
- Centralizer of $\mathcal{A}$ inside G_+ | $Z := \left\{\mathrm{diag}\left(e^{\mathrm{i}x}, e^{\mathrm{i}x}\right) \mid x \in \mathbb{R}^n\right\}$
- Projections | $\pi_1(y, Y, \upsilon^\ell, \upsilon^r) \mapsto y$, $\pi_2 \colon (y, Y, \upsilon^\ell, \upsilon^r) \mapsto Y$
- Abelian subalgebras | $\mathfrak{Q}_1 := \pi_1^*(C^\infty(G)^{G_+ \times G_+})$, $\mathfrak{Q}_2 := \pi_2^*(C^\infty(\mathcal{G})^G)$

Some standard results

1. $\forall\, y \in G$, $\exists\, y_L, y_R \in G_+$ and $\exists!\, q \in \mathbb{R}^n : \pi/2 \geq q_1 \geq \cdots \geq q_n \geq 0$ such that
$$y = y_L e^{\mathrm{i}Q(q)} y_R^{-1}.$$
Strict inequalities for q restrict freedom $(y_L, y_R) \to (y_L\zeta, y_R\zeta)$, $\forall\, \zeta \in Z$.
2. $\forall\, g \in G_-$, $\exists\, \eta \in G_+$ and $\exists!\, q \in \mathbb{R}^n : \pi/2 \geq q_1 \geq \cdots \geq q_n \geq 0$ such that
$$g = \eta e^{2\mathrm{i}Q(q)} \eta^{-1}$$
Strict inequalities for q restrict freedom $\eta \to \eta\zeta$, $\forall\, \zeta \in Z$.
3. $\forall\, Y_- \in \mathcal{G}_-$, $\exists\, g_R \in G_+$ and $\exists!\, d \in \mathbb{R}^n : d_1 \geq \cdots \geq d_n \geq 0$ such that
$$Y_- = g_R \mathrm{i}\, Q(d) g_R^{-1}$$
Strict inequalities for d restrict freedom $g_R \to g_R\zeta$, $\forall\, \zeta \in Z$.

$2n+1$ particles move symmetrically w.r.t. a fixed point Q_0 on a circle of radius $r = 1/2$. Interaction via pair potential inversely proportional to the square of the chord-distance gives the **trigonometric $\mathbf{BC}_n$ Sutherland system**.

The configuration space is a **Weyl alcove**

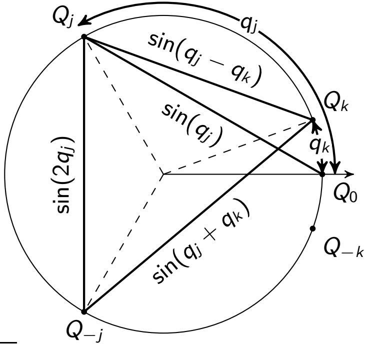
$$C_1 := \{q \in \mathbb{R}^n \mid \pi/2 > q_1 > \cdots > q_n > 0\},$$

and the phase space is its cotangent bundle

$$M := C_1 \times \mathbb{R}^n = \{(q,p) \mid q \in C_1,\ p \in \mathbb{R}^n\},$$

with the canonical symplectic form

$$\omega = \sum_{j=1}^n dq_j \wedge dp_j.$$



The dynamics is governed by the Hamiltonian

$$H(q,p) = \frac{1}{2}\sum_{j=1}^n p_j^2 + \sum_{j=1}^n \frac{\gamma_1}{\sin^2(q_j)} + \sum_{j=1}^n \frac{\gamma_2}{\sin^2(2q_j)} + \sum_{1\leq j<k\leq n}\left(\frac{\gamma}{\sin^2(q_j-q_k)} + \frac{\gamma}{\sin^2(q_j+q_k)}\right),$$

$\gamma, \gamma_1, \gamma_2$ are **real coupling constants** satisfying $\gamma > 0$, $\gamma_2 > 0$, $4\gamma_1 + \gamma_2 > 0$.

Solving the momentum constraint $J(y, Y, \upsilon^\ell \oplus \upsilon^r) = 0$ by **"diagonalizing" the group component** gives the following:

Lax matrix of Sutherland system

$$Y(q,p) := K(q,p) - \mathrm{i}\kappa C,$$

where $K(q,p)$ is the $N \times N$ matrix

$$K_{j,k} = -K_{n+j,n+k} = \mathrm{i}p_j\delta_{j,k} - \mu(1-\delta_{j,k})/\sin(q_j - q_k),$$
$$K_{j,n+k} = -K_{n+j,k} = (\nu/\sin(2q_j) + \kappa\cot(2q_j))\delta_{j,k} + \mu(1-\delta_{j,k})/\sin(q_j + q_k).$$

Theorem (analogue of L. Fehér – B.G. Pusztai on hyperbolic model, '07)

1. $S := \left\{(e^{\mathrm{i}Q(q)}, Y(q,p), \upsilon^\ell_{\mu,\nu}(V_{\mathbb{R}}), \upsilon^r_\kappa) \mid (q,p) \in M\right\} \subset P_0$ is a **global cross-section** for the action of $G_+ \times G_+$ on $P_0 = J^{-1}(0)$.
2. Identifying P_{red} with S, the reduced symplectic form is the **Darboux form** $\omega = \sum_{k=1}^n dq_k \wedge dp_k$.
3. Thus the obvious identification between S and M provides a **symplectomorphism** $(P_{\mathrm{red}}, \Omega_{\mathrm{red}}) \simeq (M, \omega)$.
4. The functional independence of the family $H_k(q,p) := \frac{1}{4k}\mathrm{tr}(-\mathrm{i}Y(q,p))^{2k}$, $k = 1, \dots, n$ implies the **Liouville integrability** of the trigonometric BC_n Sutherland system since $H_1 = H$.

At a "semi global" level the dual model has the following Hamiltonian

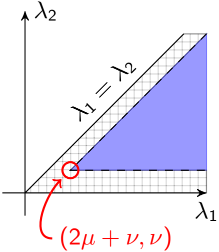
$$\tilde{H}^0(\lambda,\vartheta) = \sum_{j=1}^{n} \cos(\vartheta_j)\left[1-\frac{\nu^2}{\lambda_j^2}\right]^{\frac{1}{2}}\left[1-\frac{\kappa^2}{\lambda_j^2}\right]^{\frac{1}{2}}\prod_{\substack{k=1\\(k\neq j)}}^{n}\left[1-\frac{4\mu^2}{(\lambda_j\pm\lambda_k)^2}\right]^{\frac{1}{2}}$$

$$+\frac{\nu\kappa}{4\mu^2}\prod_{j=1}^{n}\left[1-\frac{4\mu^2}{\lambda_j^2}\right]-\frac{\nu\kappa}{4\mu^2},$$

with $\boldsymbol{\mu},\boldsymbol{\nu},\boldsymbol{\kappa}$ **real coupling constants** satisfying $\mu>0$, $\nu>|\kappa|\geq 0$.

The coordinates $\lambda_1,\dots,\lambda_n$ vary in a **Weyl chamber with thick walls**

$$C_2 = \left\{\lambda\in\mathbb{R}^n \,\middle|\, \begin{matrix}\lambda_a-\lambda_{a+1}>2\mu,\\(a=1,\dots,n-1)\end{matrix} \quad \text{and} \quad \lambda_n>\max\{\nu,|\kappa|\}\right\},$$



and $\vartheta_1,\dots,\vartheta_n$ are angular variables. The Hamiltonian $\tilde{H}^0$ generates dynamics via the symplectic form

$$\tilde{\omega}^0 = \sum_{k=1}^{n} d\lambda_k\wedge d\vartheta_k.$$

This system can be viewed as a particular real form of the complex **rational BC_n Ruijsenaars – Schneider – van Diejen system.**

Solving the constraint $J(y, Y, \upsilon^\ell \oplus \upsilon^r) = 0$ by **"diagonalizing" the Lie algebra part** gives the following:

Lax matrix of the rational BC_n RSvD system

$$L(y(\lambda,\vartheta)) = h(\lambda)\check{A}(\lambda,\vartheta)h(\lambda) = h(\lambda)\left[\frac{2\mu f_j\overline{(Cf)}_k - 2(\mu-\nu)C_{j,k}}{2\mu + \Lambda_k - \Lambda_j}\right]_{j,k} h(\lambda)$$

with $\Lambda = (\lambda, -\lambda)$ and $h(\lambda)$ is the $2n \times 2n$ unitary matrix

$$h(\lambda) = \begin{bmatrix} \alpha(\lambda) & \beta(\lambda) \\ -\beta(\lambda) & \alpha(\lambda) \end{bmatrix},$$

where the real functions $\alpha(x), \beta(x)$ are defined by the formulae

$$\alpha(x) = \frac{\sqrt{x + \sqrt{x^2 - \kappa^2}}}{\sqrt{2x}}, \quad \beta(x) = \kappa \frac{1}{\sqrt{2x}} \frac{1}{\sqrt{x + \sqrt{x^2 - \kappa^2}}}.$$

$$f_a = \left[1 - \frac{\nu}{\lambda_a}\right]^{\frac{1}{2}} \prod_{\substack{b=1 \\ (b\neq a)}}^{n} \left[1 - \frac{2\mu}{\lambda_a \pm \lambda_b}\right]^{\frac{1}{2}}, \quad f_{n+a} = e^{\mathrm{i}\vartheta_a}\left[1 + \frac{\nu}{\lambda_a}\right]^{\frac{1}{2}} \prod_{\substack{b=1 \\ (b\neq a)}}^{n} \left[1 + \frac{2\mu}{\lambda_a \pm \lambda_b}\right]^{\frac{1}{2}}$$

The diagonalization $Y \to \mathrm{i}\,\mathrm{diag}(\lambda, -\lambda)$ can be used to define the map

$$\mathfrak{L} : P_0 \to \mathbb{R}^n, \quad (y, Y, \upsilon^\ell, \upsilon^r) \mapsto \lambda.$$

Theorem (L. Fehér – TFG, ’14)

1. $\tilde{S} := \{(y(\lambda, \vartheta), \mathrm{i}h(\lambda)\Lambda(\lambda)h(\lambda)^{-1}, \upsilon^\ell_{\mu,\nu}(V(\lambda,\vartheta)), \upsilon^r) \mid (\lambda, e^{\mathrm{i}\vartheta}) \in C_2 \times \mathbb{T}^n\}$ is contained in the constraint surface $P_0 = J^{-1}(0)$ and it provides a **cross-section** for the $G_+ \times G_+$-action restricted to $\mathfrak{L}^{-1}(C_2) \subset P_0$. In particular, $C_2 \subset \mathfrak{L}(P_0)$ and $\tilde{S}$ intersects every gauge orbit in $\mathfrak{L}^{-1}(C_2)$ precisely in one point.
2. The pull-back of the symplectic form Ω is $\Omega|_{\tilde{S}} = \sum_{k=1}^n d\lambda_k \wedge d\vartheta_k$
3. The family of functions $\tilde{\mathcal{H}}_k(y, Y, \upsilon^\ell, \upsilon^r) = \frac{(-1)^k}{2k}\mathrm{tr}(y^{-1}CyC)^k$ restricted to $\tilde{S}$ become RSvD Hamiltonians, in particular for $k = 1$

$$\tilde{\mathcal{H}}_1|_{\tilde{S}}(\lambda, \vartheta) = \frac{1}{2}\mathrm{tr}\big(h(\lambda)\check{A}(\lambda,\vartheta)h(\lambda)\big) = \tilde{H}^0(\lambda, \vartheta).$$

4. Hence the Hamiltonian reduction of the system $(P, \Omega, \tilde{\mathcal{H}}_1)$ reproduces $(\tilde{M}^0, \tilde{\omega}^0, \tilde{H}^0)$ via restriction to the open **dense** submanifold $\mathfrak{L}_{\mathrm{red}}^{-1}(C_2) \subset P_{\mathrm{red}}$ identified with $C_2 \times \mathbb{T}^n$.

Introducing the complex variables

$$z_j = \sqrt{\lambda_j - \lambda_{j+1} - 2\mu} \prod_{a=1}^{j} e^{\mathrm{i}\vartheta_a},\ j = 1, \dots, n-1, \quad z_n = \sqrt{\lambda_n - \nu} \prod_{a=1}^{n} e^{\mathrm{i}\vartheta_a}$$

enables one to complete the "semi-global" model of the dual system into a global model by allowing the zero value for the complex variables $z_1, \dots, z_n$.
This completion results from the symplectic reduction automatically.

Theorem (L. Fehér – TFG, '14)

1. $\hat{S} := \{(\hat{y}(z), \mathrm{i}(h\Lambda h^{-1})(\lambda(z)), \upsilon^{\ell}_{\mu,\nu}(\hat{V}(z)), \upsilon^{r}) \mid z \in \mathbb{C}^n\}$ defines a **global cross-section** for the $G_+ \times G_+$-action on the constraint surface P_0.
2. The parametrization of the elements of $\hat{S}$ by $z \in \mathbb{C}^n$ gives rise to a **symplectic diffeomorphism** between $(P_{\mathrm{red}}, \Omega_{\mathrm{red}})$ and $\mathbb{C}^n$ equipped with the **Darboux form** $\mathrm{i}\sum_{k=1}^{n} d\bar{z}_k \wedge dz_k$.
3. The spectral invariants of the "global RSvD Lax matrix" $\hat{L}(z) \equiv h(\lambda(z))\hat{A}(z)h(\lambda(z))$ yield **commuting Hamiltonians** on $\mathbb{C}^n$ that represent the reductions of the Hamiltonians spanning the Abelian Poisson algebra $\mathfrak{Q}_1$.

The Sutherland Lax matrix is diagonalizable

$$Y(q,p) \sim \mathrm{i}\Lambda(\lambda) = \mathrm{i}\,\mathrm{diag}(\lambda, -\lambda)$$

hence the action-angle transform of Sutherland Hamiltonians are of the form

$$h_k(\lambda) = \frac{\lambda_1^{2k} + \cdots + \lambda_n^{2k}}{2k}, \quad k = 1, \dots, n$$

and have a **global minimum** in the closure of C_2

$$\min_{(q,p)\in C_1\times\mathbb{R}^n} H_k(q,p) = \min_{\lambda\in\overline{C_2}} h_k(\lambda) = h_k(\lambda^0),$$

where λ^0 is the boundary point

$$\lambda_a^0 = (n-a)2\mu + \nu, \quad a = 1, \dots, n.$$

In terms of the "oscillator variables" $z \in \mathbb{C}^n$ the **equilibrium** $(q,p) = (q^e, 0)$ of the Sutherland system corresponds to $z = 0$. Furthermore, the similarity of

$$-(h(\lambda)\check{A}(\lambda, e^{\mathrm{i}\vartheta})h(\lambda))^\dagger \quad \text{and} \quad \mathrm{diag}(\exp(\mathrm{i}q), \exp(-\mathrm{i}q))$$

along with the fact that $[h(\lambda), m] = 0$ for any $m \in Z_{G_+}(\mathcal{A})$ imply (with the special choice $m = m(e^{\mathrm{i}\vartheta})$) that

$$\sigma(-(h\hat{A}h)^\dagger(z)) = \{e^{2\mathrm{i}q_1}, \dots, e^{2\mathrm{i}q_n}, e^{-2\mathrm{i}q_1}, \dots, e^{-2\mathrm{i}q_n}\}.$$

In particular, for $z = 0$ the above spectral identity provides a useful method to determine the components of q^e.

The 'unreduced RSvD Hamiltonians' $\tilde{\mathcal{H}}_k(y, Y, V)$, $k = 1, \ldots, n$ have the action-angle transforms

$$\tilde{h}_k(q) := \frac{(-1)^k}{k} \sum_{j=1}^{n} \cos(2kq_j), \quad k = 1, \ldots, n.$$

Our aim is to show that the dual model is **maximally superintegrable** by constructing extra constants of motion of the form

$$f_i(q, p) := \sum_{j=1}^{n} p_j (X^{-1}(q))_{j,i}, \quad i \in \{1, \ldots, n\} \setminus \{k\},$$

where X is the $n \times n$ matrix

$$X_{a,b} := \frac{\partial \tilde{h}_a(q)}{\partial q_b} = (-1)^{a+1} 2 \sin(2aq_b), \quad a, b = 1, \ldots, n.$$

A simple calculation* shows that

$$\det X(q) = (-1)^{\frac{n(n+3)}{2}} 2^{\frac{n(n+1)}{2}} \prod_{j=1}^{n} \sin 2q_j \prod_{1 \le j < k \le n} (\cos 2q_k - \cos 2q_j).$$

Hence $X(q)$ is invertible at every $q \in C_1$.

A Poisson commuting family of functions involving the Hamiltonian of the rational BC_n RSvD model was found by van Diejen ('94)

$$H_\ell(\lambda,\theta) = \sum_{\substack{J\subset\{1,\dots,n\},\ |J|\leq\ell \\ \varepsilon_j=\pm1,\ j\in J}} \cosh(\theta_{\varepsilon J}) V_{\varepsilon J;J^c}^{1/2} V_{-\varepsilon J;J^c}^{1/2} U_{J^c,\ell-|J|}, \quad \ell=1,\dots,n.$$

A natural question to ask: Is there a relation between van Diejen's functions and the spectral invariants of the Lax matrix $L(\lambda,\theta)$? (This does not seem to be trivial due to superintegrability.)
Let K_m denote the coefficients of the characteristic polynomial of the Lax matrix

$$\det(L(\lambda,\theta)-x\mathbf{1}_{2n}) = K_0(\lambda,\theta)x^{2n}+K_1(\lambda,\theta)x^{2n-1}+\dots+K_{2n-1}(\lambda,\theta)x+K_{2n}(\lambda,\theta).$$

Proposition (L. Fehér – TFG, '15)

We have the following linear relation between the action-angle transforms

$$(-1)^\ell\mathcal{H}_\ell(q) = \mathcal{K}_\ell(q) + \sum_{m=0}^{\ell-1} \frac{2(n-m)}{2(n-m)-(\ell-m)} \binom{(n-\ell)+(n-m)}{\ell-m} \mathcal{K}_m(q).$$

Our argument relies on the scattering theory of the rational BC_n RSvD system.

Results

By working in the powerful geometric framework of symplectic reduction we have obtained:

- **Lax matrix** for the rational BC_n RSvD model with 3 couplings (μ, ν, κ)
- **Action-angle duality** between the trigonometric BC_n CMS and completed rational RSvD systems with a global characterization of the phase spaces
- **Equilibrium** of trigonometric BC_n CMS system
- **Superintegrability** of the derived dual system
- **Equivalence** of the two families of Hamiltonians

Related topics & future plans

- New deformation of the BC_n Sutherland system found by Marshall ('13) reduction on Poisson – Lie groups, trigonometric analogue **to appear**
- Lax matrices for the hyperbolic BC_n RSvD model **in preparation**
- New compact forms of the trigonometric RS systems, Fehér – Kluck ('13) quantization (commuting difference op's, joint eigenfunctions) **in progress**
- Integrable random matrix ensembles generated by Lax matrices of CMS/RSvD systems, Bogomolny – Giraud – Schmit ('09) **in progress**

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