The Electrostatic Properties of Zeros of Exceptional Polynomials

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Preliminaries:

- Classical orthogonal polynomials: eigenfunctions of Sturm-Liouville operators, bound-state solutions to exactly solvable potentials in quantum mechanics.

$$\cap$$

- General orthogonal polynomials: $(p_n = p_n(w))$ orthogonal on $(a, b) \subset \mathbb{R}$ w.r.t. $w = e^{-Q}$, Q : twice differentiable, convex.

$$\deg p_n = n,\ n = 0, 1, \ldots \qquad p_n \text{ has } n \text{ simple zeros in } (a, b),$$

$$p_n''(x) + M_n(x)p_n'(x) + N_n(x)p_n(x) = 0.$$

$M_n(x)$, $N_n(x)$ depend on p_n (H. Mhaskar, M. Ismail)

- Exceptional orthogonal polynomials: X_m-class: $p_{m,m+n}$, $m \geq 1$

orthogonal on $(a,b) \subset \mathbb{R}$ w.r.t. w_m

$$\deg p_{m,m+n} = m + n, \; n = 0, 1, \dots$$

$p_{m,m+n}$ has n (simple) regular zeros in (a,b),
m exceptional zeros in the exterior of (a,b)

$$p''_{m,m+n}(x) + M_n(x)p'_{m,m+n}(x) + N_n(x)p_{m,m+n}(x) = 0.$$

$M_n(x)$, $N_n(x)$ depend on m and n.

(D. Gómez-Ullate, N. Kamran, R. Milson, F. Marcellán, S. Odake, R. Sasaki, C. Quesne, D. Dimitrov, Yen Chi Lun, A. Durán, A. Kuijlaars...)

Notations:

$U_n := \{u_1, \ldots, u_n\}$: any system of nodes on an interval I, $0 \leq w \in C^2(I)$: a weight function on I,

$$\omega_{U_n}(x) := \prod_{k=1}^{n}(x - u_k).$$

The energy function on I with respect to w is

$$T_w(u_1, \ldots, u_n) = \prod_{j=1}^{n} w(u_j) \prod_{1 \leq i < j \leq n} (u_i - u_j)^2$$

The weighted Fejér constants on I with respect to U_n and w are:

$$C_k := C_{k,U_n,w} = \frac{\omega''_{U_n}}{\omega'_{U_n}}(u_k) + \frac{w'}{w}(u_k)$$

Lemma. $p_n(x) = \gamma_n \prod_{i=1}^{n}(x - \zeta_i)$, *with zeros* $Z_n = \{\zeta_1, \ldots, \zeta_n\}$,

$$p_n^{''}(x) + M_n(x)p_n^{'}(x) + N_n(x)p_n(x) = 0,$$

where

$$(\log w_n(x))^{'} = M_n(x).$$

Let $T_{w_n}(u_1, \ldots, u_n)$ *be the energy function with respect to* w_n. *Then*

$$\frac{\partial \log T_{w_n}(u_1,\ldots,u_n)}{\partial u_i}(\zeta_1, \ldots, \zeta_n) = C_{i,w_n,Z_n} = 0,$$

$$\frac{\partial^2 \log T_{w_n}(u_1,\ldots,u_n)}{\partial u_i \partial u_j}(\zeta_1, \ldots, \zeta_n) =: H_{i,j} = \frac{2}{(\zeta_i - \zeta_j)^2},$$

$$\frac{\partial^2 \log T_{w_n}(u_1,\ldots,u_n)}{\partial u_i^2}(\zeta_1, \ldots, \zeta_n) =: H_{i,i} = -\frac{2}{3}\Phi(\zeta_i),$$

where $\Phi(x) := \Phi_{w_n}(x)$ *is the coefficient of the transformed differential equation:*

$$z_n^{''}(x) + \Phi(x)z_n(x) = 0,$$

which is satisfied by

$$z_n(x) = p_n(x)\sqrt{w_n(x)}.$$

X_m-Laguerre-(I) Polynomials

$w^{(\alpha)}(x) = x^{\alpha}e^{-x}$: the Laguerre weights on $(0, \infty)$, $L_k^{(\alpha)}(x)$: the k^{th} Laguerre polynomial. The classical Laguerre differential equation:

$$xy'' + (\alpha + 1 - x)y' + ny = 0$$

The zeros of $L_k^{(\alpha)}(x)$ are $0 < \zeta_{k,1}^{\alpha} < \cdots < \zeta_{k,k}^{\alpha}$, the zeros of $L_m^{(\alpha-1)}$, are $y_1, \ldots, y_m$.

X_m-Laguerre-(I) polynomials $(m \geq 1)$[G. K. M. **JMAA**, (2009)]:

$\{L^{I,(\alpha)}_{m,m+n}\}_{n=0}^{\infty}$.

- orthogonal on $(0,\infty)$ w.r.t. $\hat{w}^{(\alpha)}_m := \frac{|x|^\alpha e^{-x}}{S^2(x)}$, where $S(x) := L^{(\alpha-1)}_m(-x)$.

- satisfies the differential equation

$$y''(x) + \left(\frac{\alpha+1-x}{x} - \frac{2S'(x)}{S(x)}\right)y'(x) + \left(\frac{m+n}{x} - \frac{\alpha}{x}\frac{2S'(x)}{S(x)}\right)y(x) = 0$$

Lemma [Gómez-Ullate, Marcellán, Milson **JMAA**, (2013)]

For $\alpha > 0$ $L^{I,(\alpha)}_{m,m+n}$ *has* m *simple exceptional zeros in* $(-\infty, 0)$*:* $z_{m,n,1} > \cdots > z_{m,n,m}$*, and* n *simple regular zeros in* $(0,\infty)$*:* $x_{m,n,1} < \cdots < x_{m,n,n}$*. The location of these zeros is the following:* $0 < x_{m,n,1} < \zeta^{\alpha}_{n,1}$*,* $\zeta^{\alpha}_{n-1,j-1} < x_{m,n,j} < \zeta^{\alpha}_{n,j}$*,* $-\zeta^{\alpha}_{m,1} < z_{m,n,1} < 0$ *and* $-\zeta^{\alpha}_{m,j} < z_{m,n,j} < -\zeta^{\alpha}_{m-1,j}$*. Futhermore*

$$\lim_{n\to\infty} n x_{m,n,j} = \frac{\left(j^{(\alpha)}_j\right)^2}{4}, \tag{1}$$

where $\{j^{(\alpha)}_j\}_{j\geq 1}$ *is the increasing sequence of the positive zeros of the Bessel function* $J_\alpha(z)$*, and the exceptional zeros of* $L^{I,(\alpha)}_{m,m+n}$ *converge to the* m *zeros of* $L^{(\alpha-1)}_m(-x)$*.*

Further Notations and Remarks:

$$L^{I,(\alpha)}_{m,m+n} = \left(\frac{1}{m!}\prod_{i=1}^{m}(x - z_i)\right)\frac{(-1)^n}{n!}\prod_{j=1}^{n}(x - x_j) = P_{m,n}(x)q_{m,n}(x)$$

$$w_n = \widehat{w}_m^{(\alpha+1)} = \frac{|x|^{\alpha+1}e^{-x}}{S^2(x)}$$

$$M_n(x) = \frac{\alpha+1-x}{x} - \frac{2S^{'}(x)}{S(x)} = (\log w_n(x))^{'}$$

$$T_{\widehat{w}_m^{(\alpha+1)}}\left(u_1, \ldots, u_{m+n}\right) = \prod_{i=1}^{m+n}\widehat{w}_m^{(\alpha+1)}(u_i)\prod_{1\le i<j\le n+m}(u_i - u_j)^2.$$

Theorem [H. **JAT**,(2015)]

Let $\alpha \geq 1$ and $n \geq 0$. Then the logarithmic energy function, $\log T_{\widehat{w}_m^{(\alpha+1)}}$ *has a saddle point at* $Z_{m,n} = \{z_{m,n,1}, \ldots, z_{m,n,m}, x_{m,n,1}, \ldots, x_{m,n,n}\}$, *which are the zeros of* $L^{I,(\alpha)}_{m,m+n}$. *More precisely* $\frac{\partial^2 \log T_{\widehat{w}_m^{(\alpha+1)}}(u_1,\ldots,u_n)}{\partial u_i^2}(Z_{m,n})$ *is positive if u_i is one of the first m variables, and it is negative if u_i is one of the last n variables.*

Remark:

For X_1-Jacobi polynomials similar theorem is proved by D. Dimitrov, Yen Chi Lun (**JAT** 2014).

Let v is a new weight on $(0,\infty)$, which depends on n:

$$v := v_{m,n}^{(\alpha+1)} := \widehat{w}_m^{(\alpha+1)} P_{m,n}^2 = \frac{x^{\alpha+1}e^{-x}P_{m,n}^2(x)}{S_m^2(x)}.$$

$$T_v(u_1,\dots,u_n) = \prod_{j=1}^n v(u_j) \prod_{1\le i<j\le n}(u_i-u_j)^2.$$

$$\frac{\partial \log T_{\widehat{w}_m^{(\alpha+1)}}(u_1,\dots,u_{n+m})}{\partial u_{m+i}}(z_1,\dots,x_n) = \frac{\partial \log T_v(u_1,\dots,u_n)}{\partial u_i}(x_1,\dots,x_n),$$

$$\frac{\partial^2 \log T_{\widehat{w}_m^{(\alpha+1)}}(u_1,\dots,u_{n+m})}{\partial u_{m+i}^2}(z_1,\dots,x_n) = \frac{\partial^2 \log T_v(u_1,\dots,u_n)}{\partial u_i^2}(x_1,\dots,x_n).$$

$$q''(x) + M_{1,n}(x)q'(x) + N_{1,n}(x)q(x) = 0,$$

$$M_{1,n}(x) = M_n(x) + 2\frac{P'_{m,n}}{P_{m,n}}(x) = (\log v)'(x)$$

$$N_{1,n}(x) = N_n(x) + \frac{P''_{m,n}}{P_{m,n}}(x) + M_n(x)\frac{P'_{m,n}}{P_{m,n}}(x).$$

Moreover $q_{m,n}(x)\sqrt{v(x)}$ fulfils the differential equation:

$$f''(x) + \Phi_1(x)f(x) = 0, \quad \text{where} \quad \Phi_1(x) = N_{1,n}(x) - \frac{1}{4}M_{1,n}^2(x) - \frac{1}{2}M'_{1,n}(x).$$

Theorem [H.**JAT**,(2015)]

Let $\alpha \geq 1$. *The positive zeros of* $L^{I,(\alpha)}_{m,m+n}$:
$X_{m,n} = \{x_{m,n,1}, \ldots, x_{m,n,n}\}$ *is the unique set of minimal energy (or Fekete set) with respect to the external field represented by the weight* $\left(v^{(\alpha+1)}_{m,n}\right)^{\frac{1}{2(n-1)}}$.

Egerváry-Turán interpolation problem:

It is to find an interpolation process of the lowest degree and of the smallest norm. Earlier results: e.g. I. Joó (1975), K. Balázs (1979), P. Rutka and R. Smarzewski (2012), H. (2013).

Definition. We denote by $\hat{l}_k(x)$ any polynomials of arbitrary degree for which $\hat{l}_k(x_i) = \delta_{ki}$, $i = 1, \ldots, n$. *Let w be a positive weight. The interpolation system of polynomials $\hat{l}_k(x)$, $k = 1, \ldots, n$ is w-stable on (a, b) if for all $y_1, \ldots, y_n \geq 0$*

$$0 \leq w(x) \sum_{k=1}^{n} \frac{\hat{l}_k(x)}{w(x_k)} y_k \leq \max_k y_k, \quad x \in (a, b).$$

A w-stable interpolatory system on (a, b) is the most economical, if

$$\sum_{k=1}^{n} \deg\left(\hat{l}_k(x)\right)$$

is the minimal. The Grünwald operator on X_n is:

$$Y_n(f, x) := \sum_{k=1}^{n} l_k^2(x) f(x_k),$$

where l_k is the fundamental polynomial of Lagrange interpolation.

Theorem [H.**JAT**,(2015)]

If $\alpha > 1$, the Grünwald operator on the positive zeros of $L_{m,m+n}^{I,(\alpha)}$ is $v_{m,n}^{(\alpha+1)}$-stable and it is also the most economical.

Since by the notations of the first lemma

$$H_n\left(\frac{1}{w_n}, \left(\frac{1}{w_n}\right)', x\right) = Y_n\left(\frac{1}{w_n}, x\right),$$

it is enough to prove

Lemma [H.**JAT**,(2015)] *If $\alpha > 1$,*

$$\left(\frac{1}{v_{m,n}^{(\alpha+1)}}\right)^{(2n)}(x) > 0, \quad x \in (0,\infty), \quad n = 0, 1, 2, \ldots.$$

n^{th} transfinite diameter and the speed of convergence

Normalizing the weight v, we can write

$$-\log\left((T_v)^{\frac{1}{n(n-1)}}\right) = \frac{2}{n(n-1)} \sum_{1\le i<j\le n} k_n(u_i,u_j) =: d(u_1,\ldots,u_n),$$

where

$$k_n(x,y) := -\log\left(\frac{c}{n}|x-y| v^{\frac{1}{2(n-1)}}(x) v^{\frac{1}{2(n-1)}}(y)\right),$$

is the modified kernel (with $\frac{c}{n}$), which is symmetric, lower semicontinuous and positive and it depends on n.

By standard arguments (e.g. H. Mhaskar, E. Saff, Where does the Sup Norm... ; V. Totik,...Varying Weight) it can be seen that

$$d_n = \inf_{u_1,\dots,u_n \in (0,\infty)} d(u_1, \dots, u_n)$$

has a limit as n tends to infinity. Here we give an estimation on the speed of convergence by classical methods.

Theorem [H.**JAT**,(2015)]

Let $\alpha \geq 1$. With the notations above $d_n = d_{n-1} + O\left(\frac{\log^2 n}{n^2}\right)$.

X_m-Jacobi Polynomials

$P_n^{(\alpha,\beta)}(x)$ are the classical Jacobi polynomials, which are orthogonal with respect to the weight $w^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$ on the interval $(-1,1)$ $(\alpha, \beta > -1)$. $P_n^{(\alpha,\beta)}$ satisfies the differential equation

$$(1-x^2)y'' + (\beta - \alpha - (\alpha+\beta+2)x)\, y' + n(n+\alpha+\beta+1)y = 0.$$

X_m-Jacobi Polynomials $(m \geq 1)$ [O. S. (**L. Phys.**, 2009)]: $\left\{ \widehat{P}_{m,m+n}^{(\alpha,\beta)} \right\}_{n=0}^{\infty}$

- orthogonal w.r.t. $\frac{w^{(\alpha,\beta)}(x)}{S^2(x)}$ on $(-1,1)$, where $S(x) := P_m^{(-\alpha-1,\beta-1)}(x)$.

- satisfy the differential equation

$$y'' + \left(\frac{\beta-\alpha-(\alpha+\beta+2)x}{1-x^2} - \frac{2S'(x)}{S(x)} \right) y' + \left(\frac{m(\alpha-\beta-m+1)+n(n+\alpha+\beta+1)}{1-x^2} - \frac{\beta}{1+x}\frac{2S'(x)}{S(x)} \right) y = 0$$

Lemma [Gómez-Ullate, Marcellán, Milson (**JMAA**, 2013)]

Let us suppose that α, β *and* m *satisfy the condition* $\alpha + 1 - m - \beta \notin \{0, 1, \ldots, m-1\}$*, and one of the conditions below*

(A) $\beta, \alpha + 1 - m \in (-1, 0)$

(B) $\beta, \alpha + 1 - m \in (0, \infty)$*.*

Then $\widehat{P}^{(\alpha,\beta)}_{m,m+n}$ *has exactly* n *regular zeros (i.e. zeros in* $(-1,1)$*) which are all simple, and* m *exceptional zeros (i.e. zeros out of* $[-1,1]$*); furthermore the regular zeros of* $\widehat{P}^{(\alpha,\beta)}_{m,m+n}$ *approach the zeros of the classical Jacobi polynomials* $P^{(\alpha,\beta)}_n$*, the exceptional zeros of* $\widehat{P}^{(\alpha,\beta)}_{m,m+n}$ *approach the zeros of* $P^{(-\alpha-1,\beta-1)}_m$*, as* n *tends to infinity.*

let $\widehat{P}^{(\alpha,\beta)}_{m,m+n} = P_{m,n} q_{m,n}$, where $q := q_{m,n}$ has the regular, $P := P_{m,n}$ has the exceptional zeros.

$v^{(\alpha+1,\beta+1)}_{m,n} = P^2_{m,n} \widehat{w}^{(\alpha+1,\beta+1)}_m$.

Theorem [H.**JAT**,(2015)]

If $\alpha > m-1$, $\beta > 0$ and if n is large enough, then the set of the regular zeros of $\widehat{P}^{(\alpha,\beta)}_{m,m+n}$ is the unique set of minimal energy with respect to the external field $\left(v^{(\alpha+1,\beta+1)}_{m,n}\right)^{\frac{1}{2(n-1)}}$.

X_m-Laguerre-(II) Polynomials

[O. S. (**L. Phys.**, 2010)]: $\{L^{II,(\alpha)}_{m,m+n}\}_{n=0}^{\infty}$, $m \geq 1$, $\alpha > m-1$.

- orthogonal on $(0,\infty)$ w.r.t. $\widehat{w}^{(\alpha)} := \frac{x^{\alpha}e^{-x}}{s^2(x)}$, where $S(x) := L_m^{(-\alpha-1)}(x)$.

- satisfies the differential equation

$$y''(x) + \left(\frac{\alpha+1-x}{x} - \frac{2S'(x)}{S(x)}\right) y'(x) + \left(\frac{n-m}{x} - \frac{\alpha}{x}\frac{2S'(x)}{S(x)}\right) y(x) = 0.$$

Lemma [Gómez-Ullate, Marcellán, Milson (**JMAA**, 2013)]

$L^{II,(\alpha)}_{m,m+n}$ has $n+m$ simple zeros, n regular zeros $x^{(\alpha)}_{m,1}, \dots, x^{(\alpha)}_{m,n} \in (0,\infty)$ and 0 or 1 negative zero according to whether m is even or odd. Furtheremore

$$\lim_{n\to\infty} n x^{(\alpha)}_{m,i} = \frac{\left(j_i^{(\alpha)}\right)^2}{4},$$

and as $n \to \infty$ the exceptionl zeros of $L^{II,(\alpha)}_{m,m+n}$ converge to the zeros of $S(x) = L_m^{(-\alpha-1)}(x)$.

$L^{II,(\alpha)}_{m,m+n} = P(x)q(x)$ ($P(x)$ is a polynomial of degree m and with 0 or 1 real zero, q has the regular zeros.)

$$v(x) = v^{(\alpha+1)}_{m,n}(x) = \frac{x^{\alpha}e^{-x}P^2(x)}{S^2(x)}$$

Theorem [H.**JAT**,(2015)]

If $\alpha > m-1$, and if n is large enough, then the set of the regular zeros of $\hat{L}m, m+n^{(II,\alpha)}$ is the unique set of minimal energy with respect to the external field $\left(v^{(\alpha+1)}_{m,n}\right)^{\frac{1}{2(n-1)}}$.

Exceptional Hermite polynomials [G-U. G. M. **J. Phys.**(2014), D. **JAT**,(2014)]

NOTATION. Partition λ of length r and of size $|\lambda|$:

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1), \ |\lambda| = \sum_{i=1}^{r} \lambda_i.$$

The generalized Hermite polynomial associated with λ is

$$H_\lambda = \mathrm{Wr}\left[H_{\lambda_r}, \ldots, H_{\lambda_2+r-2}, H_{\lambda_1+r-1}\right].$$

Wr is the Wronskian determinant.

$$\deg H_\lambda = |\lambda|.$$

λ an even partition if r is even and $\lambda_{2k-1} = \lambda_{2k}$ for $k = 1, \ldots, \frac{r}{2}$.

If λ an even partition:

H_λ has no zeros on $\mathbb{R}$. (V. Adler **Th. Math. Phys.**, (1994))

A positive weight function on $\mathbb{R}$:

$$w_\lambda = \frac{e^{-x^2}}{(H_\lambda(x))^2}$$

$$N_\lambda := \{n \geq |\lambda| - r : n \neq |\lambda| + \lambda_i - i,\ i = 1, \ldots, r\},$$

For $n \in N_\lambda$ let

$$P_n := \mathrm{Wr}\left[H_{\lambda_r}, \ldots, H_{\lambda_2+r-2}, H_{\lambda_1+r-1}, H_{n-|\lambda|+r}\right].$$

deg $P_n = n$ and P_n satisfies the differential equation (D. Gómez Ullate, Y. Grandati, R. Milson, **J. Phys. A** (2014)))

$$P_n'' + \left(-2x - 2\frac{H_\lambda'}{H_\lambda}\right) P_n' + \left(\frac{H_\lambda''}{H_\lambda} + 2x\frac{H_\lambda'}{H_\lambda} + 2n - |\lambda|\right) P_n = 0.$$

(The differential equation of the classical Hermite polynomials H_n is $y'' - 2xy' + 2ny = 0$.)

$$\int_{\mathbb{R}} P_n P_m w_\lambda = 0.$$

$\{P_n\}_{n\in N_\lambda}$ is complete in the Hilbert space $L^2(\mathbb{R}, w_\lambda)$, (A. Durán **JAT**, (2014)).

NOTATION. λ is a fixed even partition $H_\lambda = H$, $w_\lambda = w$,

$\deg H = m$. We assume that all the zeros of H, $w_k = u_k + iv_k$ are simple.

(By a conjecture of Felder, Hemery and Veselov (**Phys.** (2012)) for any partition λ the zeros of H_λ are simple, except possibly for the zero at $z = 0$. The conjecture is known in the special case $\lambda = (\nu, \nu)$ (M. Garcia-Ferrero, D. Gómez-Ullate **Lett. in Math. Phys.**, (2015)).

Z: the set of the zeros of the exceptional Hermite polynomials, $P_{m,m+n}$.

$\{z_k = \xi_k + i\eta_k, k = 1, \ldots, m\}$: the exceptional zeros of $P_{m,m+n}$;

$\{x_i, i = 1, \ldots, n\}$ are the regular (real) zeros of $P_{m,m+n}$.

The energy function:

$$\log T_w = F(\dot{\xi}_1, \dot{\eta}_1, \ldots, \dot{\xi}_m, \dot{\eta}_m, \dot{x}_1, \ldots, \dot{x}_n) = \Re F_c,$$

where

$$F_c(u_1, \ldots, u_{m+n}) = -\sum_{i=1}^{m+n} u_i^2 - 2\sum_{i=1}^{m+n} \log H(u_i) + 2\sum_{1\le i<j\le m+n} \log(u_i - u_j).$$

Remark. By the differencial equation: all the first partial derivatives of F are zero at Z.

The Hessian: $\mathbf{H} = [h_{i,j}] \in \mathbb{R}^{2m+n \times 2m+n}$.

Theorem [H.(2015)]

If n is large enough, $\mathbf{H}(Z)$ is nonsingular.

Actually we prove that $\hat{\mathbf{H}} = D^{-1}\mathbf{H}D$ is nonsingular, where D is a diagonal matrix.

Main tool: partitioned matrices.

A partition π of $\mathbb{C}^n$: a finite collection $\{W_i\}_{i=1}^{l}$ of pairwise disjoint linear subspaces, each having dimension at least unity, whose direct sum is $\mathbb{C}^n$. Furthermore let $\Phi_\pi := (\Phi_1, \ldots, \Phi_l)$, where Φ_j is a norm on W_j , for each $j = 1, \ldots, l$. $A = [A_{i,j}] \in \mathbb{C}^{n\times n}$ is strictly block diagonally dominant with respect to Φ_π if

$$\left(\|A_{ii}^{-1}\|_{\Phi_\pi}\right)^{-1} > \sum_{\substack{1\le j\le l\\ j\ne i}} \|A_{ij}\|_{\Phi_\pi}, \ \ 1 \le i \le l.$$

Here matrix norm: (l_∞-norm) $A \in \mathbb{C}^{n\times m}$, then $\|A\| = \max_{i=1}^{n} \left(\sum_{j=1}^{m} |a_{i,j}|\right)$.

Lemma [R. Varga (2004, Springer)]

Given a partition π of $\mathbb{C}^n$ and given Φ_π, assume that $A = [A_{i,j}] \in \mathbb{C}^{n\times n}$, partitioned by π, is strictly block diagonally dominant with respect to Φ_π. Then, A is nonsingular.

$$A_{k,k} = \begin{bmatrix} F''_{\dot{\xi}_k\dot{\xi}_k}(Z) & F''_{\dot{\eta}_k\dot{\xi}_k}(Z) \\ F''_{\dot{\xi}_k\dot{\eta}_k}(Z) & F''_{\dot{\eta}_k\dot{\eta}_k}(Z) \end{bmatrix}$$

Lemma

There is a positive constant c such that

$$\|A_{k,k}^{-1}\|^{-1} > cn.$$

$$\sum_{\substack{1 \le l \le m \\ l \ne k}} \max\left\{\left|h_{2k-1,2l-1}\right| + \left|h_{2k-1,2l}\right| ; \left|h_{2k,2l-1}\right| + \left|h_{2k,2l}\right|\right\}$$

$$+ \sum_{i=1}^{n} \max\left\{\left|h_{2k-12m+i}\right| ; \left|h_{2k2m+i}\right|\right\} = O(\sqrt{n})$$

The Gerschgorin set with respect to the partition π and the norm Φ_π is $G_\pi^{\Phi_\pi}(A) = \cup_{i=1}^{l} G_{i,\pi}^{\Phi_\pi}(A)$, where $G_{i,\pi}^{\Phi_\pi}(A) := \left\{ z \in \mathbb{C} : \left(\|(zI_i - A_{i,i})^{-1}\|_{\Phi_\pi} \right)^{-1} \leq \sum_{\substack{1 \leq j \leq l \\ j \neq i}} \|A_{ij}\|_{\Phi_\pi} \right\}$, $(1 \leq i \leq l)$, where I_i denotes the identity matrix for the subspace W_i. $\sigma(A)$ is the spectrum of A.

$G = G_r \cup G_e$, where G_r is generated by the regular zeros and G_e is generated by the exceptional zeros. Since $\mathbf{H}$ is symmetric, the eigenvalues of $\hat{\mathbf{H}}$ are real, thus we mean $G_r = G_r \cap \mathbb{R}$, and $G_e = G_e \cap \mathbb{R}$.

Theorem [H.(2015)]

Let n be large enough. $G_r \subset \mathbb{R}_-$, and if $x \in G_r$ then $|x| \leq cn$. If $x \in G_e$, then $|x| \sim n$.

Lemma [R. Varga (2004, Springer)] Given a partition π of $\mathbb{C}^n$ and given Φ_π, let $A = [A_{i,j}] \in \mathbb{C}^{nn}$, partitioned by π. If $\lambda \in \sigma(A)$, there is an $i \in \{1, \ldots, l\}$ such that $\lambda \in G_{i,\pi}^{\Phi_\pi}(A)$. That is $\sigma(A) \subset G_\pi^{\Phi_\pi}(A)$.

Special case:

$$H = d_\nu := \begin{vmatrix} h_\nu & h_{\nu+1} \\ h'_\nu & h'_{\nu+1} \end{vmatrix}.$$

Theorem [H.(2015)]

Let $H = d_\nu$. If n is large enough, the logarithmic energy function $F = \log T_w$ has a "saddle point" at the set of zeros of orthogonal polynomials with respect to w, more precisely the modified Hessian $\hat{\mathbf{H}}$ is diagonally dominant, $h_{2m+i2m+i} < 0$, $i = 1, \ldots, n$ and $h_{2l-1,2l-1} = -h_{2l,2l} < 0$, $l = 1, \ldots, m$.

Lemmas

(1) For $\nu = 0, 1, \ldots,$ d_ν fulfils the following differential equation

$$d_\nu'' - \left(2x + 2\frac{h_\nu'}{h_\nu}\right) d_\nu' + 4x\frac{h_\nu'}{h_\nu} d_\nu = 0.$$

(2)

$$h_{2k-1,2k-1} = F''_{\dot\xi_k \dot\xi_k}(Z) = -h_{2k,2k} = F''_{\dot\eta_k \dot\eta_k}(Z) = \Re r_{m,n}(z_k),$$

$$h_{2k-1,2k} = F''_{\dot\eta_k \dot\xi_k}(Z) = \Im r_{m,n}(z_k),$$

where

$$r_{m,n}(x) = -\left(8\left(x^2 + 1 - \sqrt{\frac{\nu+1}{2}}\frac{2h_\nu^2 + h_{\nu+1}^2}{d_\nu}\right) + 2n\right).$$

(3) Let H, $P_{m,m+n}$ be the same as above, w_k, z_k $(k = 1, \ldots, m)$ the zeros of H and the exceptional zeros of $P_{m,m+n}$ respectively. If n is large enough, there is a $c > 0$ such that for $k = 1, \ldots, m$ $|z_k - w_k| \geq \frac{c}{\sqrt{n}\log n}$.

$P_{m,m+n} = P_m q_n$, where $P_m = \prod_{k=1}^{m}(x - z_k)$, $z_k = \xi_k + i\eta_k$ depends on n.

Theorem [H.(2015)]

If n is large enough, the logarithmic energy function with respect to the weight $w_1 := wP_m^2$ attains its maximum at a unique set, which is the set of the regular zeros $x_1, \ldots, x_n$ of the exceptional Hermite polynomial $P_{m,m+n}$.

Remark. q_n satisfies

$$y'' + \left(-2x - 2\frac{H'}{H} + 2\frac{P_m'}{P_m}\right) y'$$

$$+ \left(\frac{P_m''}{P_m} + \frac{P_m'}{P_m}\left(-2x - 2\frac{H'}{H}\right) + \frac{H''}{H} + 2x\frac{H'}{H} + 2n - m\right) y = 0.$$