Week 8

Numerical methods for physicists, 2018/19 autumn semester

Solution of nonlinear equations - `fzero, fsolve, fminunc`

PROBLEM 1. In 1225, Fibonacci gave the only real zero of the polynomial $p(x) = x^3 + 2x^2 + 10x - 20$ to 9 decimal places as $x^\star = 1.368808107$. The exact value is

$$x^\star = \frac{1}{3}\left(\sqrt{27}\,\sqrt{5240} + 352\right)^{1/3} - \frac{26/3}{\left(\sqrt{27}\,\sqrt{5240} + 352\right)^{1/3}} - \frac{2}{3}.$$

Show that p has exactly one zero in the interval [0,1]. Approximate the zero with the

a) bisection method (only a few steps, how many step we need to achieve an error of 10^{-6}?),

b) Newton's method (choose an appropriate starting value, give an error estimate after the fourth step, observe the order of the convergence),

c) fixed point iteration using the fixed point reformulation

$$x = \frac{20}{x^2 + 2x + 10}$$

(check the assumptions of the Banach fixed point theorem, observe the order of the convergence).

d) with the `fsolve` command of Matlab.

PROBLEM 2. Try to solve the equation $p(x) := x^3 - 5x^2 + 8x - 4 = 0$ with the Newton's method starting from the point $x_0 = 3$. The method will find the root $x^\star = 2$. Determine the convergence order of the sequence obtained. How could we get back the second order convergence of the Newton's method?

PROBLEM 3. We would like to compute the zero nearest to 1 of the function

$$f(x) = \frac{x}{8}(63x^4 - 70x^2 + 15).$$

By the use of Matlab, let us show that the iteration formula

$$x_{k+1} = x_k - f(x_k)/10 =: F(x_k)$$

can be used to obtain the zero of the function on the interval $[0.8, 1]$ (check the assumptions of the Banach fixed point theorem). Execute the iteration with $x_0 = 1$ and compute the zero in question of the function to 6 correct decimal places.

PROBLEM 4. We compute the root $x^\star = 2$ of the equation

$$\frac{2x^2 - 3x - 2}{x - 1} = 0$$

using the iteration

$$x_{k+1} = x_k - 1 + \frac{1}{x_k - 1}.$$

What is the order of the convergence? Where can we start the iteration?

PROBLEM 5. Find a solution of the system

$$\begin{aligned} x^2 + 2y^2 - y - 2z &= 0 \\ x^2 - 8y^2 + 10z &= 0 \\ x^2 - 7yz &= 0 \end{aligned}$$

near the point $(1, 1, 1)$. Use Newton's method and Matlab's built-in solver.

PROBLEM 6. Find a local minimizer of the function $f(x_1, x_2) = (1 - x_1)^2 + 5(x_2 - x_1^2)^2$ around the point (2,2). Use Matlab's `fminunc` function.

HOMEWORK FOR WEEK 8 - to be submitted until the next computer lab (The detailed solutions can be submitted either on A4 sheets of paper (printed or written) or in a pdf file (e.g. in an exported Matlab livescript) to rhorvath@math.bme.hu. Do not send Matlab files. Answer all questions with a sentence at the end of each problem.)

1. (2p) Show that the function $F(x) = (1 + \sin x)/2$ has a unique fixed point in the interval $[0, 1]$. Compute the fixed point with Matlab to 6 correct decimal places. What is the order of convergence of the produced sequence?

2. (2p) Find the root of the equation $x^5 - x - 2 = 0$ using Newton's method. (Try to find an appropriate initial value to start the iteration, and execute the iteration in Matlab.)

3. (2p) Find a solution of the system

$$\begin{aligned} x + y + z &= 0 \\ x^2 + y^2 + z^2 &= 2 \\ x(y + z) &= -1 \end{aligned}$$

near the point $(3/4, 1/2, -1/2)$. Use Newton's method and Matlab's built-in solver.