Weeks 3-4

Numerical methods for physicists, 2018/19 autumn semester

Strictly diagonally dominant matrices, SPD matrices, M-matrices

PROBLEM. 1. Show that the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

is an M-matrix. Give an upper bound for its inverse in maximum norm.

PROBLEM. 2. Show that the matrix in Problem 1 is symmetric positive definite.

PROBLEM. 3. Show that if the diagonal of a matrix is positive, the offdiagonal is nonpositive, and the matrix is strictly diagonally dominant, then the matrix is an M-matrix.

Conditioning of linear systems (Condition number of a matrix, sensibility of the solution of a SLAEs to the coefficients)

PROBLEM. 4. Give an upper bound in maximum and 1-norms for the condition number of the matrix $\mathbf{A}$ in Problem 1. Compute the condition numbers with Matlab.

PROBLEM. 5. Let

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 3 \end{bmatrix}$$

and let $\tilde{\mathbf{A}}$ be an arbitrary 4×4 matrix such that $\max_{i,j} |a_{ij} - \tilde{a}_{ij}| \leq 0.02$. Give an upper estimate for the relative change of the solution of the system $\mathbf{A}\overline{\mathbf{x}} = [1, 1, 1, 1]^T$ if we change the matrix $\mathbf{A}$ to $\tilde{\mathbf{A}}$. Let us suppose that we know that $\|\mathbf{A}^{-1}\|_\infty = 2.5$. Check the result in Matlab.

PROBLEM. 6. The vector $\overline{\mathbf{r}} = \overline{\mathbf{b}} - \mathbf{A}\overline{\mathbf{y}}$ is called the residual vector of the system $\mathbf{A}\overline{\mathbf{x}} = \overline{\mathbf{b}}$ for the fixed vector $\overline{\mathbf{y}}$. Compute the residual vectors of the system

$$\begin{bmatrix} 34 & 55 \\ 55 & 89 \end{bmatrix} \overline{\mathbf{x}} = \begin{bmatrix} 21 \\ 34 \end{bmatrix}$$

for the vectors

$$\overline{\mathbf{x}}_1 = \begin{bmatrix} -0.11 \\ 0.45 \end{bmatrix}, \quad \overline{\mathbf{x}}_2 = \begin{bmatrix} -0.99 \\ 1.01 \end{bmatrix}.$$

What do you think which vector approximates better the exact solution $\overline{\mathbf{x}}^\star$ of the system? Construct an upper bound for the $\|\overline{\mathbf{x}}_i - \overline{\mathbf{x}}^\star\|$ error using the norm of the residual vector.

Solution of linear systems with the Gaussian method, LU decomposition

PROBLEM. 7. Solve the system

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ -1 & 5 & 8 \end{bmatrix} \overline{\mathbf{x}} = \begin{bmatrix} 4 \\ 14 \\ 17 \end{bmatrix}$$

with the Gaussian elimination method. Give the LU decomposition of the coefficient matrix.

PROBLEM. 8. Count the number of operations of the Gauss-Jordan solution method of linear systems (we eliminate elements both below and above the diagonal). Compare the result with the number of the operations of the Gaussian method. (Gauss-Jordan method is useful for inverting matrices, operation count $2n^3 + O(n^2)$)

HOMEWORK FOR WEEKS 3-4 - to be submitted until the next computer lab (The detailed solutions can be submitted either on A4 sheets of paper (printed or written) or in a pdf file (e.g. in an exported Matlab livescript) to rhorvath@math.bme.hu. Do not send Matlab files. Answer all questions with a sentence at the end of each problems.)

1. (2p) Write the Matlab code that solves the system

$$\begin{aligned} \varepsilon x_1 + x_2 &= 1 \\ x_1 + x_2 &= 0 \end{aligned}$$

using the Gaussian method with $\varepsilon = 10^{-16}$. What is the solution given by Matlab? The exact solution should be

$$\overline{\mathbf{x}}^\star = \left[\frac{1}{\varepsilon - 1}, \frac{-1}{\varepsilon - 1}\right]^T.$$

Explain the difference of the exact and the computed solutions (see Slide 27).

2. (1p) Give the LU decomposition of the matrix

$$\begin{bmatrix} 4 & 1 & -2 \\ 1 & 17/4 & 1/6 \\ -2 & 1/6 & 19/9 \end{bmatrix}$$

manually.