Week 2

Numerical methods for physicists, 2018/19 autumn semester

Norms and eigenvalues

PROBLEM. 1. Show that the relation $\varrho(\mathbf{A}) \le \|\mathbf{A}\|$ is valid for all submultiplicative matrix norms. (Frobenius norm is a submultiplicative norm.)

PROBLEM. 2. Give an upper bound for the spectral radius of the matrix

$$\mathbf{A} = \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.25 \end{bmatrix}.$$

Is it true that $\varrho(\mathbf{A}) < 1$? Give the sum of the series (if it exists) $\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \ldots$.

PROBLEM. 3. Show that if $\|\mathbf{A}\| < 1$ in some induced norm then $\mathbf{I} - \mathbf{A}$ is nonsingular and we have the estimate (see Thm. 9)

$$\|(\mathbf{I} - \mathbf{A})^{-1}\| \le \frac{1}{1 - \|\mathbf{A}\|}.$$

Order of convergence

PROBLEM. 4. Let us consider the two sequences generated by the iterations

$$x_{k+1} = 2 - 2 \cdot \frac{2 - x_k}{4 - x_k^2}, \quad y_{k+1} = y_k - \frac{y_k^2 - 2}{y_k + y_{k-1}}$$

where $x_1 = 1$, and $y_1 = 1$, $y_2 = 2$. Both sequences tend to $\sqrt{2}$. Determine the order of the convergence of the sequences.

Condition numbers, conditioning (Conditioning of a problem, conditioning of a computation, well-conditioned and ill-conditioned problems)

PROBLEM. 5. Let us consider the problem $x - \sqrt{1 + d} + 1 = 0$, where d is the data and x is the solution.

Show that if d is close to 0, then the (relative) condition number of the problem is close to 1. How does the problem behave from the point of view of conditioning?

The condition number for $d = 3.4 \times 10^{-16}$ is $\kappa(d) = 0.99999999\ldots$. In this case we have $x_d = 1.69999999\ldots \times 10^{-16}$, moreover with $\delta d = -0.1 \times 10^{-16}$ we have $x_{d+\delta d} = 1.64999999 \times 10^{-16}$. Check the meaning of the condition number with the above data.

Use Matlab to calculate the values x_d and $x_{d+\delta d}$ with the algorithm: d is given, $a = 1 + d$, $b = \sqrt{a}$, $x = b - 1$. What can be said about the conditioning of the above numerical calculation.

Repeat the above investigation with the algorithm: d is given, $a = 1 + d$, $b = \sqrt{a}$, $c = 1 + b$, $x = 1/c$.

PROBLEM. 6. Let us consider the linear system $\mathbf{A}\overline{\mathbf{x}} - \overline{\mathbf{d}} = \overline{\mathbf{0}}$, where $\mathbf{A}$ is a real quadratic matrix, the vector $\overline{\mathbf{d}}$ is the data and the vector $\overline{\mathbf{x}}$ is the solution. Give an upper bound for the condition number of the problem. What can be said about the conditioning of the above problem if $\mathbf{A}$ is orthogonal?

Floating point numbers (Representation of real numbers, machine epsilon, machine precision, double precision numbers of Matlab)

PROBLEM. 7. A primitive calculator uses decimal floating point numbers with a 1 digit long mantissa and the characteristics that falls into the interval $[-2, 2]$. Compute the value of the quantities $1/3$, $1/900$, $20 \cdot 200$, $2 + 0.1 + 0.1 + \ldots + 0.1$ (group the sums from left to right and vice versa).

PROBLEM. 8. Compute the value of $\exp(-25)$ with Matlab using the partial sums of the Taylor series of the exponential function. How can we improve the result?

PROBLEM. 9. Compute the value of the expression $1/(\sqrt{x^2+1} - x)$ with Matlab for $x = 10^8$. Explain the result. How can we improve the result?

PROBLEM. 10. Compute the value of the expression $((1 + x) - 1)/x$ with Matlab for $x = 10^{-15}$. Explain the result. How can we improve the result?

PROBLEM. 11. Compute the absolute and relative errors of computer divisions.

PROBLEM. 12. How would you compute the values of the expressions $\sqrt{1+x} - \sqrt{1-x}$ (for values close to zero) and $\log\sqrt{x+1} - \log\sqrt{x}$ (for large values)?

HOMEWORK FOR WEEK 2 - to be submitted until the next computer lab (The detailed solutions can be submitted either on an A4 sheet of paper (printed or written) or in a pdf file (e.g. in an exported Matlab livescript) to rhorvath@math.bme.hu. Do not send Matlab files. Answer all questions with a sentence at the end of each problems.)

1. (2p) Let us apply the difference quotient

$$d_i = \frac{f(x_0 + h_i) - f(x_0)}{h_i}$$

to approximate the derivative of the function $f(x) = \sin^2(x)$ at the point $x_0 = \pi/4$ (exact value is 1). Let us use the values $h_i = 0.1/2^i$ $(i = 1, \ldots, 15)$. Compute the errors of the approximation for all h_i values, and use the graphical method to estimate the order of the convergence.

2. (1p) How does the conditioning of the problem $x^3 + d - 1 = 0$ (d is the data and x is the solution) depend on the data d?