Week 1

Numerical methods for physicists, 2018/19 autumn semester

Special matrices (Band matrix, diagonal matrix, upper and lower triangular matrices, upper and lower Hessenberg matrices, tridiagonal matrix, symmetric and skew-symmetric matrix, orthogonal matrix, permutation matrix, definiteness, diagonally dominant matrices)

PROBLEM. 1. Construct the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 5 & 6 & 0 & 0 & 0 \\ 2 & 3 & 5 & 6 & 0 & 0 \\ 1 & 2 & 3 & 5 & 6 & 0 \\ 0 & 1 & 2 & 3 & 5 & 6 \\ 0 & 0 & 1 & 2 & 3 & 5 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{bmatrix}.$$

Display the nonzero element structure of the matrix.

PROBLEM. 2. Construct the tridiagonal matrix

$$\mathbf{B} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \in \mathbb{R}^{20\times 20}.$$

How much memory is used to store this matrix? Define the same matrix taking advantage of its sparse structure. Define the identity matrix as a sparse matrix.

PROBLEM. 3. Define a random 6×6 matrix $\mathbf{C}$ and define the upper and lower triangular parts and the diagonal part of this matrix. Define the upper Hessenberg part of the matrix.

PROBLEM. 4. Show that the product of two upper triangular matrices is also upper triangular. Show that the inverse of an upper triangular matrix is upper triangular again.

PROBLEM. 5. Show that the matrix is orthogonal.

$$\mathbf{B} = \begin{bmatrix} 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}.$$

Vector and matrix norms (definitions $(p, 1, 2, \infty)$, properties of the norms, induced matrix norms)

PROBLEM. 6. Show that in case of $p \to \infty$, the p-norm of a vector tends to its maximum norm.

Problem. 7. Draw the 1-radius neighborhood of the origin in the plane in 1, 2 and maximum norms. Are there two vectors, say $\overline{\mathbf{a}}$ and $\overline{\mathbf{b}}$, such that $\|\overline{\mathbf{a}}\|_* \leq \|\overline{\mathbf{b}}\|_*$ and $\|\overline{\mathbf{b}}\|_{**} \leq \|\overline{\mathbf{a}}\|_{**}$ in two appropriate norms?

Problem. 8. Compute the 1, 2 and ∞ norms of the vector $\overline{\mathbf{x}} = [1, -2, 1, 3, -4]^T$ with Matlab and manually. Notice that $\|\overline{\mathbf{x}}\|_2^2 = \overline{\mathbf{x}}^T\overline{\mathbf{x}}$ for all vectors.

Problem. 9. Show that the maximum and the Euclidean norms are equivalent. Compute the equivalence constants.

Problem. 10. Show that

$$\|\mathbf{A}\| := \sup_{\overline{\mathbf{x}} \neq \mathbf{0}} \frac{\|\mathbf{A}\overline{\mathbf{x}}\|}{\|\overline{\mathbf{x}}\|}$$

defines a matrix norm (induced matrix norm).

Problem. 11. Show the consistency, sub-multiplicativity properties of an induced matrix norm. What is the norm of the identity matrix in an induced matrix norm?

Problem. 12. Show that the p-norm of a diagonal matrix is the maximum absolute value in the diagonal.

Problem. 13. Show that the multiplication with orthogonal matrices does not change the 2-norm of a matrix or a vector.

Problem. 14. Show for one of the vector norms 1 or ∞ the formula for the induced matrix norm.

Problem. 15. Compute the 1, ∞ and 2-norms of the matrix with Matlab and manually (only the first two)

$$\mathbf{C} = \begin{bmatrix} 2 & -1 & 1 & 1 & 1 \\ -1 & 2 & -4 & 2 & 6 \\ -3 & 3 & 2 & -5 & 4 \\ 6 & 7 & -1 & 4 & -9 \\ 4 & -3 & 0 & -3 & 3 \end{bmatrix}.$$