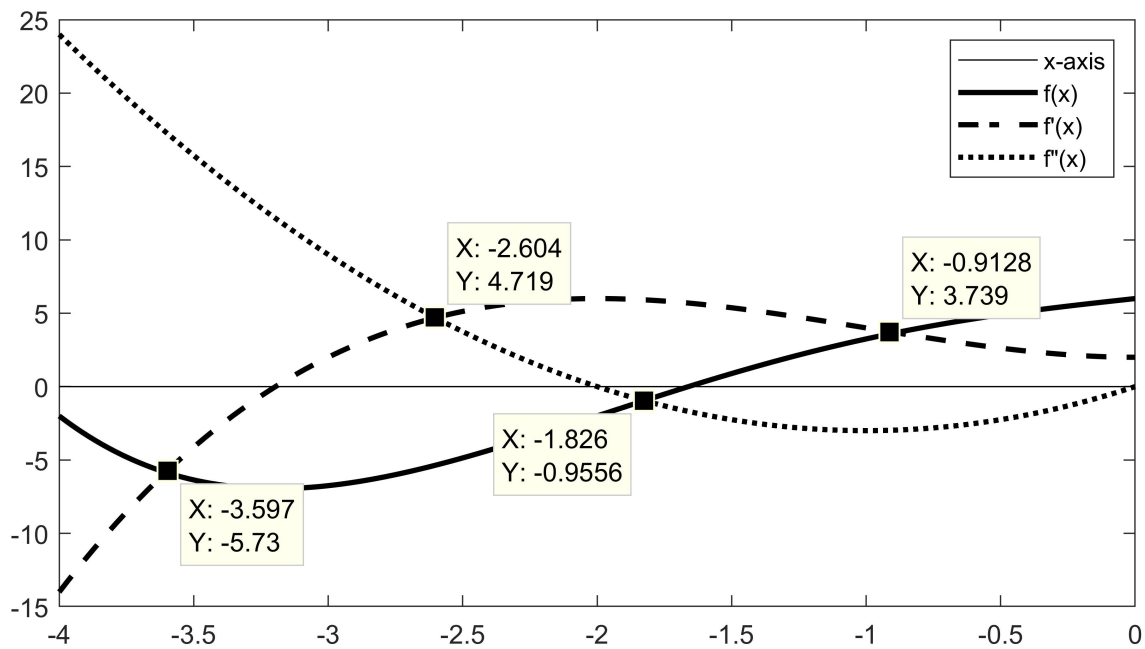
Numerical methods, midterm test II (2018/19 autumn, A)

PROBLEM 1. (6p) We would like to approximate the zero nearest to -1.5 of the function $f(x) = x^4/4 + x^3 + 2x + 6$ using Newton's method. Let us consider the possible initial points $x_0 = -3.5$, $x_0 = -2.5$, $x_0 = -1.8$, $x_0 = -1.5$ or $x_0 = 0$. One of these points ensures monotone convergence. Choose this initial point to start the method and give an estimate to the zero of the function after two iteration steps. (The graph of the function and the derivatives can be seen in the figure.)



PROBLEM 2. (2+5p) Let $\mathbf{A} = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 2 & 3 & 2 & 0 \\ 0 & 2 & 3 & 2 \\ 0 & 0 & 2 & 3 \end{bmatrix}$. Show that all eigenvalues of the matrix are real and located in the interval $[-1, 7]$.

We are searching for the nearest eigenvalue to 1. Use an appropriate iterative method to find this eigenvalue (perform only one step with the method starting at the vector $[-2, 1, 1, -2]^T$ and then give an estimation to the corresponding eigenvalue).

PROBLEM 3. (3+4p) We have determined the natural cubic spline interpolation $s(x)$ of the points $(0,1)$, $(1,2)$, $(2,1)$ and $(3,2)$. We obtained

$$s(x) = \begin{cases} -2x^3/3 + 5x/3 + 1, & x \in [0,1] \\ s_2(x), & x \in [1,2] \\ -2x^3/3 + 6x^2 - 49x/3 + 15, & x \in [2,3]. \end{cases}$$

Give the polynomial $s_2(x)$ on the interval $[1,2]$ using Hermite–Fejér interpolation.
We interpolate $s(x)$ on the nodes: $x_0 = 0$, $x_1 = 3$. Give an upper bound for the interpolation error on the interval $[0,3]$.

PROBLEM 4. (7p) We are going to solve the equation $2x = \tan x$. Show that the sequence produced by the iteration $x_{k+1} = \arctan(2x_k)$ converges to the solution of the equation for any initial point from the interval $[1, 1.5]$. Give an estimate for the number of steps that are needed to approximate the solution within the error tolerance 10^{-6}. Let the starting point be $x_0 = 1$.

PROBLEM 5. (3+1+3p) Apply the composite midpoint rule to give an estimation to the definite integral $\int_1^4 x \log x \, \mathrm{d}x$ (exact value is 7.3404) using three equidistant subintervals.

Estimate the error we may expect when we would use 6 subintervals.

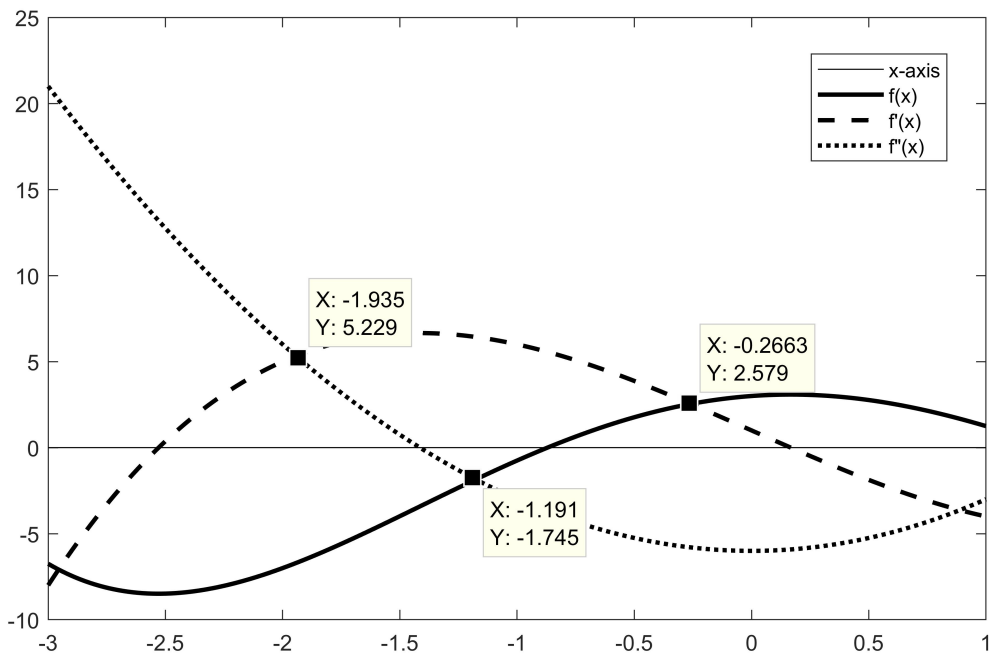
How many subintervals should we use to estimate the exact integral value within the error tolerance 10^{-4}?

PROBLEM 6. (4+2p) Give a suitable trigonometric interpolation polynomial to the points $(0,3)$, $(2\pi/3, 1)$, $(4\pi/3, 2)$.

Assume that the above points were taken from the graph of a three times continuously differentiable function f. Using the centered difference formula on the given points, give an approximation to the value $f'(4\pi/3)$.

Numerical methods, midterm test II (2018/19 autumn, B)

PROBLEM 1. (6p) We would like to approximate the zero nearest to -1 of the function $f(x) = x^4/4 - 3x^2 + x + 3$ using Newton's method. Let us consider the possible initial points $x_0 = -2.5$, $x_0 = -1.8$, $x_0 = -1.2$, $x_0 = -0.5$ or $x_0 = 0.5$. One of these points ensures monotone convergence. Choose this initial point to start the method and give an estimate to the zero of the function after two iteration steps. (The graph of the function and the derivatives can be seen in the figure.)



PROBLEM 2. (2+5p) Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$. Show that all eigenvalues of the matrix are real and located in the interval $[-3, 5]$.

We are searching for the nearest eigenvalue to -1. Use an appropriate iterative method to find this eigenvalue (perform only one step with the method starting at the vector $[2, -1, -1, 2]^T$ and then give an estimation to the corresponding eigenvalue).

PROBLEM 3. (3+4p) We have determined the natural cubic spline interpolation $s(x)$ of the points $(0,2)$, $(1,1)$, $(2,2)$ and $(3,1)$. We obtained

$$s(x) = \begin{cases} 2x^3/3 - 5x/3 + 2, & x \in [0,1] \\ s_2(x), & x \in [1,2] \\ 2x^3/3 - 6x^2 + 49x/3 - 12, & x \in [2,3]. \end{cases}$$

Give the polynomial $s_2(x)$ on the interval $[1,2]$ using Hermite–Fejér interpolation.
We interpolate $s(x)$ on the nodes: $x_0 = 0$, $x_1 = 3$. Give an upper bound for the interpolation error on the interval $[0,3]$.

PROBLEM 4. (7p) We are going to solve the equation $3x = \tan x$. Show that the sequence produced by the iteration $x_{k+1} = \arctan(3x_k)$ converges to the solution of the equation for any initial point from the interval $[1,2]$. Give an estimate for the number of steps that are needed to approximate the solution within the error tolerance 10^{-5}. Let the starting point be $x_0 = 2$.

PROBLEM 5. (3+1+3p) Apply the composite midpoint rule to give an estimation to the definite integral $\int_2^5 x \log x \, \mathrm{d}x$ (exact value is 13.4817) using three equidistant subintervals.

Estimate the error we may expect when we would use 6 subintervals.

How many subintervals should we use to estimate the exact integral value within the error tolerance 10^{-5}?

PROBLEM 6. (4+2p) Give a suitable trigonometric interpolation polynomial to the points $(0,2)$, $(2\pi/3,1)$, $(4\pi/3,3)$.

Assume that the above points were taken from the graph of a three times continuously differentiable function f. Using the centered difference formula on the given points, give an approximation to the value $f'(4\pi/3)$.