Numerical methods, midterm test II, (autumn 2017/18)
Solutions

1. Feladat. (2+2+2p) Let us consider the matrix $\mathbf{A} = \begin{bmatrix} 20 & 1 & 1 \\ 1 & -10 & 2 \\ 1 & 2 & 5 \end{bmatrix}$! Show that the power method is applicable to this matrix. Execute one iteration step with the method starting from the vector $\overline{\mathbf{x}}_0 = [100, 4, 7]^T$. Give an estimate to the single dominant eigenvalue and the corresponding eigenvector.

Solution: The matrix is symmetric. Based on the Gershgorin theorem, the three real eigenvalues are in the intervals: [-13,-7], [2,8], [18,22]. Thus, the eigenvalue with the largest absolute value is single indeed.

To execute one step, it is enough to calculate the product $\overline{\mathbf{x}}_1 = \mathbf{A}\overline{\mathbf{x}}_0 = [2011, 74, 143]^T$. This is already an estimation to the eigenvector that belongs to the dominant eigenvalue. The dominant eigenvalue can be approximated by the Rayleigh coefficient:
$(\overline{\mathbf{x}}_1^T \mathbf{A}\overline{\mathbf{x}}_1)/(\overline{\mathbf{x}}_1^T\overline{\mathbf{x}}_1) = 20.1091$.

2. Feladat. (2+4+1p) We would like to compute the zero of the function $f(x) = x - e^{-x}$ in the interval $[0, 2]$. Show that the zero is unique. Let us start with the bisection method and perform as many iteration steps that is enough to start the Newton method from the new iteration point to achieve monotone convergence to the zero of f. Execute one step with the Newton method and give an upper estimate for the error after this step.

Solution: $f'(x) = 1 + e^{-x} > 0$, that is the function is strictly monotonically increasing on $\mathbb{R}$. $f(0) < 0$ and $f(2) > 0$, that is the zero is unique in [0,2]. $f''(x) = -e^{-x} < 0$, thus the Newton method must be started from a point where the function value is negative (these are from the zero to the left). From this point the convergence will be monotone because the derivatives do not change sign. Thus we have to continue the bisection method until we arrive at a point where the function value is negative. Then we can switch to the Newton method.

Bisection method: Starting interval [0,2], $x_0 = 1$, $f(x_0) > 0$. The new interval [0,1], $x_1 = 1/2$, $f(x_1) < 0$, thus the Newton method can be started here. One step with the Newton method: $x_2 = x_1 - f(x_1)/f'(x_1) = 0.5663$.

The distance $|x_2 - x^\star|$ can be estimated as

$$|x_2 - x^\star| \leq \frac{f(x_2)}{\min_{x\in[x_2,x^\star]} |f'(x)|} \leq \frac{f(x_2)}{1 + 1/e} = 9.5366 \times 10^{-4}.$$

3. Feladat. (2+2+3p) Give the polynomial $p(x)$ and the trigonometric polynomial $t(x)$ with the least degree possible that interpolate the points

$(0,1)$, $(2\pi/3,0)$, $(4\pi/3,0)$, $(2\pi,1)$. Give an upper estimate for the value $\max_{x\in[0,2\pi]}|p(x)-t(x)|$!

Solution: We have 4 points, thus we can fit a polynomial with degree at most 3. This polynomial can be given for example by Largange's method (this is the simplest one).

$$p(x) = 1\cdot\frac{(x-2\pi/3)(x-4\pi/3)(x-2\pi)}{(0-2\pi/3)(0-4\pi/3)(0-2\pi)}+1\cdot\frac{(x-0)(x-2\pi/3)(x-4\pi/3)}{(2\pi-0)(2\pi-2\pi/3)(2\pi-4\pi/3)}$$

(the other characteristic Lagrange polynomials are multiplied by zero, this is why these terms do not appear in the above formula.)

When we fit a trigonometric polynomial, we must use only the first three points (because of the 2π-periodicity). A first degree trigonometric polynomial will do the job (the coefficients are calculated with the formula we learnt): $t(x) = 1/3 + 2\cos(x)/3$.

The maximal difference of the two functions can be calculated after noticing the fact that $p(x)$ interpolates the function $t(x)$ at the 4 given nodes. We have to estimate the interpolation error with the choice $n = 3$ (we have 4 points, of which distance is $h = 2\pi/3$) and $f(x) = t(x)$. By the use of the error formula

$$\max_{x\in[0,2\pi]}|p(x)-t(x)| \le \frac{M_4h^4}{16} = \frac{(2/3)(2\pi/3)^4}{16} = 0.8017.$$

Here $M_4 = 2/3$ is an upper estimation for the absolute values of the fourth derivative of $t(x)$.

4. Feladat. (6p) We interpolate the points $(0,0)$, $(1,1)$, $(2,3)$ with a natural cubic spline. We obtained the values $d_0 = 3/4$, $d_1 = 3/2$, $d_2 = 9/4$ for the derivatives in the interpolation nodes! Give the expression of the interpolating function in the interval $[1,2]$.

Solution: We use Hermite–Fejér interpolation:

$$s(x) = 1 + \frac{3}{2}(x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{4}(x-1)^2(x-2).$$

5. Feladat. (3+4p) Let us approximate the integral $\int_{-1}^{1} e^x/\sqrt{1-x^2}\,\mathrm{d}x$ in two ways: with the composite midpoint rule using 4 subintervals and with the two-point Gauss–Chebyshev quadrature (the weights are: $\pi/2, \pi/2$)! (For your information: the exact integral is 3.9775.)

Solution: Let $f(x) = e^x$ and $g(x) = e^x/\sqrt{1-x^2}$. The integral can be approximated by the composite midpoint rule as follows:

$$I \approx 1/2\cdot(g(-3/4)+g(-1/4)+g(1/4)+g(3/4)) = 3.0226.$$

To compute the Gauss–Chebyshev approximation we need the two zeros of the second degree Chebyshev polynomial. These are: $c_0 = -1/\sqrt{2}$ and $c_1 = 1/\sqrt{2}$. Thus the approximation:

$$I \approx (\pi/2) \cdot (f(c_0) + f(c_1)) = 3.9603.$$

It is important that the first approximation uses the values of the function g, while in the second one we use f .

6. FELADAT. (5+1+1p) We solve the initial value problem $y' = (1+x^2)y$, $y(0) = 1$ using the implicit Euler method on the interval $[0, 0.2]$. Compute the approximate solution value at the point $x = 0.2$ using the step-size $h = 0.1$. Give the exact error of the approximation provided we know the exact solution $y(x) = \exp(x + x^3/3)$. Guess the error in the case we if we used the step size $h = 0.05$.

Solution: The scheme of the implicit Euler method is $y_{k+1} = y_k + hf(x_{k+1}, y_{k+1})$. Let us apply it to the differential equation: $y_{k+1} = y_k + h(1 + x_{k+1}^2)y_{k+1}$, thus we get

$$y_{k+1} = \frac{y_k}{1 - h(1 + x_{k+1}^2)}.$$

Using the initial value $x_0 = 0$, $y(0) = 1$, and the step size $h = 0.1$ we get $y_1 = y_0/(1 - 0.1(1 + 0.1^2)) = 1.1123$, and similarly $y_2 = 1.2415$. The error at the endpoint of the interval is 0.0168. With the half of the original step size we expect half the error, that is the error value 0.0084 (the actual error is 0.0079).