Numerical methods, midterm test II. (autumn 2017/18)

1. PROBLEM. (2+2+2p) Let us consider the matrix $\mathbf{A} = \begin{bmatrix} 20 & 1 & 1 \\ 1 & -10 & 2 \\ 1 & 2 & 5 \end{bmatrix}$! Show that the power method is applicable to this matrix. Execute one iteration step with the method starting from the vector $\overline{\mathbf{x}}_0 = [100, 4, 7]^T$. Give an estimate to the single dominant eigenvalue and the corresponding eigenvector.

2. PROBLEM. (2+4+1p) We would like to compute the zero of the function $f(x) = x - e^{-x}$ in the interval [0, 2]. Show that the zero is unique. Let us start with the bisection method and perform as many iteration steps that is enough to start the Newton method from the new iteration point to achieve monotone convergence to the zero of f. Execute one step with the Newton method and give an upper estimate for the error after this step.

3. PROBLEM. (2+2+3p) Give the polynomial p(x) and the trigonometric polynomial t(x) with the least degree possible that interpolate the points $(0,1), (2\pi/3,0), (4\pi/3,0), (2\pi,1)$. Give an upper estimate for the value $\max_{x \in [0,2\pi]} |p(x) - t(x)|!$

4. PROBLEM. (6p) We interpolate the points (0,0), (1,1), (2,3) with a natural cubic spline. We obtained the values $d_0 = 3/4$, $d_1 = 3/2$, $d_2 = 9/4$ for the derivatives in the interpolation nodes! Give the expression of the interpolating function in the interval [1,2].

5. PROBLEM. (3+4p) Let us approximate the integral $\int_{-1}^{1} e^x / \sqrt{1-x^2} \, dx$ in two ways: with the composite midpoint rule using 4 subintervals and with the two-point Gauss-Chebyshev quadrature (the weights are: $\pi/2, \pi/2$)! (For your information: the exact integral is 3.9775.)

6. PROBLEM. (5+1+1p) We solve the initial value problem $y' = (1+x^2)y$, y(0) = 1 using the implicit Euler method on the interval [0, 0.2]. Compute the approximate solution value at the point x = 0.2 using the step-size h = 0.1. Give the exact error of the approximation provided we know the exact solution $y(x) = \exp(x + x^3/3)$. Guess the error in the case we if we used the step size h = 0.05.