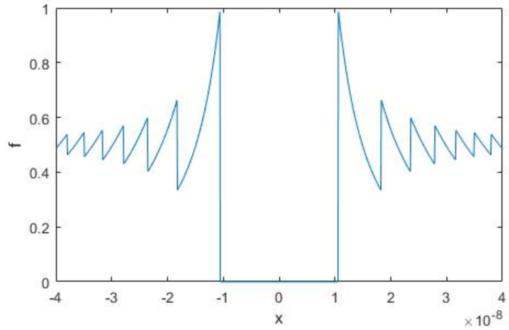
Numerical methods, midterm test I (2019/20 autumn, group A)

PROBLEM 1. (6p)

We have plotted the graph of the function $f(x) = (1 - \cos x)/x^2$ with Matlab on the interval $[-4 \times 10^{-8}, 4 \times 10^{-8}]$. The obtained graph can be seen in the figure. Let us explain the form of the graph and suggest a different form of the function that results in the correct graph. (Remember that $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$.)



PROBLEM 2. (6p) Let us decide whether a linear system with the coefficient matrix `A=-ones(6)+10*eye(6)` (9s in the diagonal, other elements are -1s) can be solved by the following methods: a) Cholesky, b) Gauss, c) relaxed Jacobi with $\omega = 1/2$, d) relaxed Gauss–Seidel with $\omega = 3$, e) conjugate gradient method. (We may use the known fact that symmetric M-matrices are positive definite matrices.)

PROBLEM 3. We solve the linear system $Ax = b$, where A is the matrix from the previous problem and b is a random vector with elements between 0.5 and 1. The solution of the system is denoted by x_1. Then we modify the system as follows. We add 0.001 to each element of the matrix A and subtract 0.01 from each element of b. The solution of the new system is denoted by x_2. Give an upper estimation for the relative error $\|x_1 - x_2\|_\infty / \|x_1\|_\infty$!

PROBLEM 4. (7p) Let us give the LU and Cholesky decompositions of the matrix (if they exist) and compute the value of $\det(A)$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 5 & 3 & 3 \\ 1 & 3 & 6 & 4 \\ 1 & 3 & 4 & 4 \end{bmatrix}!$$

PROBLEM 5. (7p) Use the relaxed Jacobi method with relaxation parameter 1/2 to solve the linear system

$$\begin{aligned} 6x_1 - x_2 &= 1 \\ -x_1 + 3x_2 - x_3 &= 2 \\ -x_2 + 2x_3 &= 1. \end{aligned}$$

Construct the iteration and estimate the number of iterations needed to approximate the exact solution of the system within the absolute error tolerance 10^{-6} in maximum norm. We start the iteration from the zero vector.

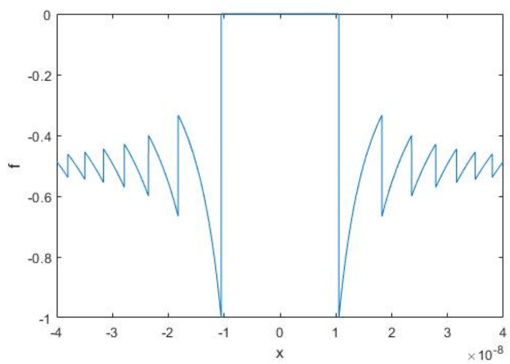
PROBLEM 6. (7p) Somebody has carried out three iteration steps for the system $Ax = b$, where A=[1,0,1;0,2,3;1,3,10] and b=[0;1;0], either with the gradient method or with the conjugate gradient method. The obtained iteration vector is $x_3 = [3/38, 14/19, -3/20]^T$ and the residual is $r_3 = [27/380, -9/380, -300/380]^T$. Let us decide which method was used and calculate the next iteration step.

Numerical methods, midterm test I (2019/20 autumn, group B)

Problem 1. (6p) Let us decide whether a linear system with the coefficient matrix `A=-ones(6)+11*eye(6)` (10s in the diagonal, other elements are -1s) can be solved by the following methods: a) Gauss, b) Cholesky, c) relaxed Gauss–Seidel with $\omega = 4$, d) relaxed Jacobi with $\omega = 1/3$, e) conjugate gradient method. (We may use the known fact that symmetric M-matrices are positive definite matrices.)

Problem 2. (6p)

We have plotted the graph of the function $f(x) = (\cos x - 1)/x^2$ with Matlab on the interval $[-4 \times 10^{-8}, 4 \times 10^{-8}]$. The obtained graph can be seen in the figure. Let us explain the form of the graph and suggest a different form of the function that results in the correct graph. (Remember that $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$.)



Problem 3. (7p) We solve the linear system $Ax = b$, where A is the matrix from Problem 1 and b is a random vector with elements between 0.3 and 1. The solution of the system is denoted by x_1. Then we modify the system as follows. We subtract 0.01 from each element of the matrix A and add 0.001 to each element of b. The solution of the new system is denoted by x_2. Give an upper estimation for the relative error $\|x_1 - x_2\|_\infty / \|x_1\|_\infty$!

Problem 4. (7p) Let us give the LU and Cholesky decompositions of the matrix (if they exist) and compute the value of $\det(A)$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 5 & 3 & 3 \\ 1 & 3 & 3 & 3 \\ 1 & 3 & 3 & 7 \end{bmatrix}!$$

Problem 5. (7p) Use the relaxed Jacobi method with relaxation parameter 1/3 to solve the linear system

$$\begin{aligned} 5x_1 - x_2 &= 1 \\ -x_1 + 4x_2 - x_3 &= 3 \\ -x_2 + 2x_3 &= 1. \end{aligned}$$

Construct the iteration and estimate the number of iterations needed to approximate the exact solution of the system within the absolute error tolerance 10^{-4} in maximum norm. We start the iteration from the zero vector.

Problem 6. (7p) Somebody has carried out three iteration steps for the system $Ax = b$, where A=[1,0,1;0,2,2;1,2,10] and b=[0;1;0], either with the gradient method or with the conjugate gradient method. The obtained iteration vector is $x_3 = [1/18, 11/18, -1/10]^T$ and the residual is $r_3 = [2/45, -1/45, -5/18]^T$. Let us decide which method was used and calculate the next iteration step.