Numerical methods, midterm test I (2018/19 autumn, group A)

PROBLEM 1. (6p) We are going to approximate the limit $\displaystyle\lim_{x\to 1}\frac{x^{3/2}-x}{\sqrt{x}-1}$ by substituting $x=0.99$ into the fraction in the present form. We use a calculator that uses a decimal number system with 2-digit-long mantissa (there is no restriction to the characteristic). Calculate the value of the fraction, explain the result, and give a better way of the calculation.

PROBLEM 2. (6p) Let us consider the two linear systems

$$a)\quad \begin{bmatrix}6 & 1\\ 1 & 8\end{bmatrix}\overline{\mathbf{x}}=\begin{bmatrix}1\\2\end{bmatrix}, \qquad b)\quad \begin{bmatrix}6 & 7\\ 7 & 8\end{bmatrix}\overline{\mathbf{x}}=\begin{bmatrix}1\\2\end{bmatrix}.$$

Which of the two systems is the most sensitive to the change of the coefficients? For that system, give an upper bound for the change of the solution of the system in 1-norm in the case when we add real numbers that are not greater than 0.02 in absolute value to each element of the coefficient matrix and to the right-hand-side vector.

PROBLEM 3. (7p) Show that if $\mathbf{A}$ is an M-matrix then the matrix $\mathbf{C}=(1/\omega)\mathbf{D}-\mathbf{R}$ is also an M-matrix for all $\omega\in(0,1]$ (here $\mathbf{A}=\mathbf{D}-\mathbf{L}-\mathbf{R}$ is the usual splittig of $\mathbf{A}$ in the iterative methods). Show that if $\overline{\mathbf{g}}$ is a majorizing vector of $\mathbf{A}$ then it is valid the estimation

$$\|\mathbf{C}^{-1}\|_\infty\le\frac{\|\overline{\mathbf{g}}\|_\infty}{\min_i(\mathbf{A}\overline{\mathbf{g}})_i}.$$

PROBLEM 4. (7p) The upper triangular matrix of the Cholesky decomposition of a matrix $\mathbf{A}$ has the form

$$\mathbf{F}=\begin{bmatrix}1 & 1 & 1 & 2\\ 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

Compute the determinant of $\mathbf{A}$, give the LU decomposition of $\mathbf{A}$ and the solution of the system $\mathbf{A}\overline{\mathbf{x}}=[3,4,5,9]^T$. Decide whether we can use the relaxed Gauss–Seidel method with relaxation parameter $\omega=1.9$ to solve this system.

PROBLEM 5. (7p) Use the Jacobi method to solve the linear system

$$\begin{aligned}6x_1-x_2&=1\\ -x_1+3x_2-x_3&=2\\ -x_2+2x_3&=1.\end{aligned}$$

Construct the iteration and estimate the number of iterations needed to approximate the exact solution of the system within the absolute error tolerance 10^{-6} in 1-norm. We start the iteration from the zero vector.

PROBLEM 6. (7p) We are going to give the QR decomposition of the matrix $\mathbf{A}=\begin{bmatrix}0 & 1\\ 1 & 1\\ 0 & \sqrt{3}\\ 0 & 0\end{bmatrix}$ using Householder reflections. The first Householder reflection, which belongs to the first column, is the matrix $\mathbf{H}_1=\begin{bmatrix}0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$. Give the QR decomposition of $\mathbf{A}$ and solve the over-determined system $\mathbf{A}\overline{\mathbf{x}}=[0,0,1,1]^T$ by the use of the QR decomposition.

Numerical methods, midterm test I (2018/19 autumn, group B)

PROBLEM 1. (6p) We are going to approximate the limit $\displaystyle\lim_{x\to 1}\frac{x-x^{3/2}}{1-\sqrt{x}}$ by substituting $x = 0.999$ into the fraction in the present form. We use a calculator that uses a decimal number system with 3-digit-long mantissa (there is no restriction to the characteristic). Calculate the value of the fraction, explain the result, and give a better way of the calculation.

PROBLEM 2. (6p) Let us consider the two linear systems

$$a)\quad \begin{bmatrix}7 & 6\\ 6 & 5\end{bmatrix}\overline{\mathbf{x}} = \begin{bmatrix}2\\1\end{bmatrix},\qquad b)\quad \begin{bmatrix}7 & 1\\ 1 & 5\end{bmatrix}\overline{\mathbf{x}} = \begin{bmatrix}2\\1\end{bmatrix}.$$

Which of the two systems is the most sensitive to the change of the coefficients? For that system, give an upper bound for the change of the solution of the system in 1-norm in the case when we add real numbers that are not greater than 0.01 in absolute value to each element of the coefficient matrix and to the right-hand-side vector.

PROBLEM 3. (7p) Show that if $\mathbf{A}$ is an M-matrix then the matrix $\mathbf{B} = (1/\omega)\mathbf{D} - \mathbf{L}$ is also an M-matrix for all $\omega \in (0,1]$ (here $\mathbf{A} = \mathbf{D} - \mathbf{L} - \mathbf{R}$ is the usual splittig of $\mathbf{A}$ in the iterative methods). Show that if $\overline{\mathbf{g}}$ is a majorizing vector of $\mathbf{A}$ then it is valid the estimation

$$\|\mathbf{B}^{-1}\|_\infty \le \frac{\|\overline{\mathbf{g}}\|_\infty}{\min_i(\mathbf{A}\overline{\mathbf{g}})_i}.$$

PROBLEM 4. (7p) The upper triangular matrix of the Cholesky decomposition of a matrix $\mathbf{A}$ has the form

$$\mathbf{F} = \begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 1 & 2\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

Compute the determinant of $\mathbf{A}$, give the LU decomposition of $\mathbf{A}$ and the solution of the system $\mathbf{A}\overline{\mathbf{x}} = [2,3,3,4]^T$. Decide whether we can use the relaxed Gauss–Seidel method with relaxation parameter $\omega = 0.1$ to solve this system.

PROBLEM 5. (7p) Use the Jacobi method to solve the linear system
$$\begin{aligned}5x_1 - x_2 &= 1\\ -x_1 + 4x_2 - x_3 &= 3\\ -x_2 + 2x_3 &= 1.\end{aligned}$$
Construct the iteration and estimate the number of iterations needed to approximate the exact solution of the system within the absolute error tolerance 10^{-4} in 1-norm. We start the iteration from the zero vector.

PROBLEM 6. (7p) We are going to give the QR decomposition of the matrix $\mathbf{A} = \begin{bmatrix}0 & \sqrt{3}\\ 0 & 1\\ 1 & 1\\ 0 & 0\end{bmatrix}$ using Householder reflections. The first Householder reflection, which belongs to the first column, is the matrix $\mathbf{H}_1 = \begin{bmatrix}0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$. Give the QR decomposition of $\mathbf{A}$ and solve the over-determined system $\mathbf{A}\overline{\mathbf{x}} = [1,0,0,1]^T$ by the use of the QR decomposition.