Numerical methods, midterm test I (2017/18. autumn semester

1. (6p) Let $\|.\|$ denote an arbitrary vector norm or the matrix norm induced by this vector norm. Prove the following statement for a quadratic matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$: If there exists a number $\alpha > 0$ such that $\|\mathbf{A}\overline{\mathbf{x}}\| \geq \alpha\|\overline{\mathbf{x}}\|$ for all vectors $\overline{\mathbf{x}} \in \mathbb{R}^n$, then $\mathbf{A}$ is non-singular and the estimate $\|\mathbf{A}^{-1}\| \leq 1/\alpha$ is valid!

2. (6p) We would like to compute the value $f(x) = e^x - x - 1$ for $x_0 = 0.0005$ as accurately as we can. We use a computer that uses decimal numbers with 8-digit-long mantissas. Let us apply first the approximation $e^{0.0005} \approx 1.0005001$! Give a better approximation of $f(x_0)$ on the same computer!

3. Show that if M is an M-matrix, then $\mathbf{AD}$ is also an M-matrix for all diagonal matrices $\mathbf{D}$ with positive diagonal. Using this statement, give an upper estimation for the maximum norm of the inverse of the matrix

$$\mathbf{B} = \begin{bmatrix} 9 & -1 & 0 \\ -3 & 3 & -3 \\ 0 & -1 & 9 \end{bmatrix} = \begin{bmatrix} 3\cdot 3 & -1 & 3\cdot 0 \\ 3\cdot(-1) & 3 & 3\cdot(-1) \\ 3\cdot 0 & -1 & 3\cdot 3 \end{bmatrix}.$$

4. We would like to solve the linear system $\mathbf{A}\overline{\mathbf{x}} = \overline{\mathbf{b}}$, where $\mathbf{A} \in \mathbb{R}^{5\times 5}$ and $\overline{\mathbf{b}} = [1,2,3,4,5]^T$. We know that $\|\mathbf{A}^{-1}\|_\infty = 4$ and that the LU decomposition of $\mathbf{A}$ has the concise form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}.$$

Compute the solution $\overline{\mathbf{x}}$ of the system. How much would the solution change in maximum norm if we changed the vector $\overline{\mathbf{b}}$ to the vector $\overline{\mathbf{b}}' = [0.99, 1.99, 3.05, 4.02, 5.1]^T$? (Hint: Do not compute matrix $\mathbf{A}$ explicitly! Let us estimate the unknown quantities.)

5. We solve system
$$\begin{aligned} 5x_1 - x_2 &= 7 \\ -x_1 + 3x_2 - x_3 &= 4 \\ -x_2 + 2x_3 &= 5 \end{aligned}$$
with an iterative solver. Show that the Gauss–Seidel method is applicable to this system. We start the iteration from the vector $\overline{\mathbf{x}}^{(0)} = [0,0,0]^T$, and we get the vector $\overline{\mathbf{x}}^{(1)} = [1.4000, 1.8000, 3.4000]^T$ in the first step (rounded to 4 decimal places). How many iterations does the iteration need to approximate the solution with a maximum error of 5×10^{-6} in 1-norm (the inverse of a lower triangular matrix is lower triangular!).

6. (7p) Give the QR decomposition of the matrix
$$\mathbf{A} = \begin{bmatrix} 0 & -4 \\ 0 & 0 \\ -5 & -2 \end{bmatrix}$$
(it is enough to use two Givens rotations)!