Problems

for the course Applied Numerical Methods with Matlab, 2019/20 spring semester

Special matrices (Band matrix, diagonal matrix, upper and lower triangular matrices, upper and lower Hessenberg matrices, tridiagonal matrix, symmetric and skew-symmetric matrix, orthogonal matrix, permutation matrix, definiteness, diagonally dominant matrices)

PROBLEM. 1. Construct the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 5 & 6 & 0 & 0 & 0 \\ 2 & 3 & 5 & 6 & 0 & 0 \\ 1 & 2 & 3 & 5 & 6 & 0 \\ 0 & 1 & 2 & 3 & 5 & 6 \\ 0 & 0 & 1 & 2 & 3 & 5 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{bmatrix}.$$

Display the nonzero element structure of the matrix.

PROBLEM. 2. Construct the tridiagonal matrix

$$\mathbf{B} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \in \mathbb{R}^{20\times 20}.$$

How much memory is used to store this matrix? Define the same matrix taking advantage of its sparse structure. Define the identity matrix as a sparse matrix.

PROBLEM. 3. Define a random 6×6 matrix $\mathbf{C}$ and define the upper and lower triangular parts and the diagonal part of this matrix. Define the upper Hessenberg part of the matrix.

PROBLEM. 4. Show that the product of two upper triangular matrices is also upper triangular. Show that the inverse of an upper triangular matrix is upper triangular again.

PROBLEM. 5. Show that the matrix is orthogonal.

$$\mathbf{B} = \begin{bmatrix} 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}.$$

Vector and matrix norms (definitions ($p, 1, 2, \infty$), properties of the norms, induced matrix norms)

PROBLEM. 6. Show that in case of $p \to \infty$, the p-norm of a vector tends to its maximum norm.

PROBLEM. 7. Draw the 1-radius neighborhood of the origin in the plane in 1, 2 and maximum norms. Are there two vectors, say $\overline{\mathbf{a}}$ and $\overline{\mathbf{b}}$, such that $\|\overline{\mathbf{a}}\|_* \leq \|\overline{\mathbf{b}}\|_*$ and $\|\overline{\mathbf{b}}\|_{**} \leq \|\overline{\mathbf{a}}\|_{**}$ in two appropriate norms?

PROBLEM. 8. Compute the 1, 2 and ∞ norms of the vector $\overline{\mathbf{x}} = [1, -2, 1, 3, -4]^T$ with Matlab and manually. Notice that $\|\overline{\mathbf{x}}\|_2^2 = \overline{\mathbf{x}}^T\overline{\mathbf{x}}$ for all vectors.

PROBLEM. 9. Show that

$$\|\mathbf{A}\| := \sup_{\overline{\mathbf{x}} \neq \mathbf{0}} \frac{\|\mathbf{A}\overline{\mathbf{x}}\|}{\|\overline{\mathbf{x}}\|}$$

defines a matrix norm (induced matrix norm).

PROBLEM. 10. Show the consistency, sub-multiplicativity properties of an induced matrix norm. What is the norm of the identity matrix in an induced matrix norm?

PROBLEM. 11. Show that the p-norm of a diagonal matrix is the maximum absolute value in the diagonal.

PROBLEM. 12. Show that the multiplication with orthogonal matrices does not change the 2-norm of a matrix or a vector.

PROBLEM. 13. Compute the 1, ∞ and 2-norms of the matrix with Matlab and manually (only the first two)

$$\mathbf{C} = \begin{bmatrix} 2 & -1 & 1 & 1 & 1 \\ -1 & 2 & -4 & 2 & 6 \\ -3 & 3 & 2 & -5 & 4 \\ 6 & 7 & -1 & 4 & -9 \\ 4 & -3 & 0 & -3 & 3 \end{bmatrix}.$$

Norms and eigenvalues

PROBLEM. 14. Show that the relation $\varrho(\mathbf{A}) \leq \|\mathbf{A}\|$ is valid for all submultiplicative matrix norms. (Frobenius norm is a submultiplicative norm.)

PROBLEM. 15. Give an upper bound for the spectral radius of the matrix

$$\mathbf{A} = \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.25 \end{bmatrix}.$$

Is it true that $\varrho(\mathbf{A}) < 1$? Give the sum of the series (if it exists) $\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \ldots$.

Order of convergence

PROBLEM. 16. Let us consider the two sequences generated by the iterations

$$x_{k+1} = 2 - 2 \cdot \frac{2 - x_k}{4 - x_k^2}, \quad y_{k+1} = y_k - \frac{y_k^2 - 2}{y_k + y_{k-1}}$$

where $x_1 = 1$, and $y_1 = 1$, $y_2 = 2$. Both sequences tend to $\sqrt{2}$. Determine the order of the convergence of the sequences.

Condition numbers, conditioning **(Conditioning of a problem, conditioning of a computation, well-conditioned and ill-conditioned problems)**

PROBLEM. 17. Let us consider the problem $x - \sqrt{1+d} + 1 = 0$, where d is the data and x is the solution.

Show that if d is close to 0, then the (relative) condition number of the problem is close to 1. How does the problem behave from the point of view of conditioning?

The condition number for $d = 3.4 \times 10^{-16}$ is $\kappa(d) = 0.99999999\ldots$. In this case we have $x_d = 1.69999999\ldots \times 10^{-16}$, moreover with $\delta d = -0.1 \times 10^{-16}$ we have $x_{d+\delta d} = 1.64999999 \times 10^{-16}$. Check the meaning of the condition number with the above data.

Use Matlab to calculate the values x_d and $x_{d+\delta d}$ with the algorithm: d is given, $a = 1+d$, $b = \sqrt{a}$, $x = b-1$. What can be said about the conditioning of the above numerical calculation.

Repeat the above investigation with the algorithm: d is given, $a = 1+d$, $b = \sqrt{a}$, $c = 1+b$, $x = 1/c$.

Floating point numbers (Representation of real numbers, machine epsilon, machine precision, double precision numbers of Matlab)

PROBLEM. 18. A primitive calculator uses decimal floating point numbers with a 1 digit long mantissa and the characteristic that falls into the interval $[-2, 2]$. Compute the value of the quantities $1/3$, $1/900$, $20 \cdot 200$, $2 + 0.1 + 0.1 + \ldots + 0.1$ (group the sums from left to right and vice versa).

PROBLEM. 19. Compute the value of $\exp(-25)$ with Matlab using the partial sums of the Taylor series of the exponential function. How can we improve the result?

PROBLEM. 20. Compute the value of the expression $1/(\sqrt{x^2+1} - x)$ with Matlab for $x = 10^8$. Explain the result. How can we improve the result?

PROBLEM. 21. Compute the value of the expression $((1+x) - 1)/x$ with Matlab for $x = 10^{-15}$. Explain the result. How can we improve the result?

PROBLEM. 22. Compute the absolute and relative errors of computer divisions.

PROBLEM. 23. How would you compute the values of the expressions $\sqrt{1+x} - \sqrt{1-x}$ (for values close to zero) and $\log\sqrt{x+1} - \log\sqrt{x}$ (for large values)?

Strictly diagonally dominant matrices, SPD matrices, M-matrices

PROBLEM. 24. Show that the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

is an M-matrix. Give an upper bound for its inverse in maximum norm.

PROBLEM. 25. Show that the matrix in Problem 24 is symmetric positive definite.

PROBLEM. 26. Show that if the diagonal of a matrix is positive, the offdiagonal is nonpositive, and the matrix is strictly diagonally dominant, then the matrix is an M-matrix.

Conditioning of linear systems (Condition number of a matrix, sensibility of the solution of a SLAEs to the coefficients)

PROBLEM. 27. Give an upper bound in maximum and 1-norms for the condition number of the matrix $\mathbf{A}$ in Problem 24. Compute the condition numbers with Matlab.

PROBLEM. 28. Let

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 3 \end{bmatrix}$$

and let $\tilde{\mathbf{A}}$ be an arbitrary 4×4 matrix such that $\max_{i,j} |a_{ij} - \tilde{a}_{ij}| \leq 0.02$. Give an upper estimate for the relative change of the solution of the system $\mathbf{A}\overline{\mathbf{x}} = [1, 1, 1, 1]^T$ if we change the matrix $\mathbf{A}$ to $\tilde{\mathbf{A}}$. Let us suppose that we know that $\|\mathbf{A}^{-1}\|_\infty = 2.5$. Check the result in Matlab. Give a similar estimate if the elements of the right-hand side vector is allowed to change with values $|\delta b_i| \leq 0.03$!

PROBLEM. 29. The vector $\overline{\mathbf{r}} = \overline{\mathbf{b}} - \mathbf{A}\overline{\mathbf{y}}$ is called the residual vector of the system $\mathbf{A}\overline{\mathbf{x}} = \overline{\mathbf{b}}$ for the fixed vector $\overline{\mathbf{y}}$. Compute the residual vectors of the system

$$\begin{bmatrix} 34 & 55 \\ 55 & 89 \end{bmatrix} \overline{\mathbf{x}} = \begin{bmatrix} 21 \\ 34 \end{bmatrix}$$

for the vectors

$$\overline{\mathbf{x}}_1 = \begin{bmatrix} -0.11 \\ 0.45 \end{bmatrix}, \quad \overline{\mathbf{x}}_2 = \begin{bmatrix} -0.99 \\ 1.01 \end{bmatrix}.$$

What do you think which vector approximates better the exact solution $\overline{\mathbf{x}}^\star$ of the system? Construct an upper bound for the $\|\overline{\mathbf{x}}_i - \overline{\mathbf{x}}^\star\|$ error using the norm of the residual vector.

Solution of linear systems with the Gaussian method, LU decomposition

PROBLEM. 30. Solve the system

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ -1 & 5 & 8 \end{bmatrix} \overline{\mathbf{x}} = \begin{bmatrix} 4 \\ 14 \\ 17 \end{bmatrix}$$

with the Gaussian elimination method. Give the LU decomposition of the coefficient matrix.

PROBLEM. 31. Count the number of operations of the Gauss-Jordan solution method of linear systems (we eliminate elements both below and above the diagonal). Compare the result with the number of the operations of the Gaussian method. (Gauss-Jordan method is useful for inverting matrices, operation count $2n^3 + O(n^2)$)

Partial pivoting - `lu`

PROBLEM. 32. Solve the system

$$\begin{aligned} 10^{-5}x + y &= 1 \\ x + y &= 2 \end{aligned}$$

with the Gaussian method without and with partial pivoting using decimal floating point numbers with four-digit-long mantissas. Compare the results. Give the general LU decomposition of the coefficient matrix in exact arithmetic.

Cholesky decomposition - `chol`

PROBLEM. 33. The LU decomposition of the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 17 & 13/2 \\ 1 & 13/2 & 23/2 \end{bmatrix}$$

is given in concise form

$$\begin{bmatrix} 4 & 2 & 1 \\ 1/2 & 16 & 6 \\ 1/4 & 3/8 & 9 \end{bmatrix}.$$

Give the Cholesky decomposition of $\mathbf{A}$ if it exists.

PROBLEM. 34. Give the Cholesky decomposition of the matrix $\mathbf{A}$ in Problem 33 directly, that is without computing the LU decomposition.

PROBLEM. 35. Solve the system $\mathbf{A}\overline{\mathbf{x}} = [3, -9/2, -21/2]^T$ ($\mathbf{A}$ is the matrix from Problem 33) using the Cholesky decomposition of the matrix.

Iterative solutions of linear systems (Jacobi and Gauss-Seidel iterations and their relaxed versions)

PROBLEM. 36. Solve the system

$$\begin{bmatrix} -1 & 5 & -2 \\ 1 & 1 & -4 \\ 4 & -1 & 2 \end{bmatrix} \overline{\mathbf{x}} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$$

with an appropriate (!) iterative solver. Write the computer code in Matlab. Estimate the number of the necessary iteration steps if we would like to achieve an error of 10^{-9} in maximum norm and the iteration starts at $\overline{\mathbf{x}}_0 = [0, 0, 0]^T$.

PROBLEM. 37. Which iterative solvers can be used to solve the system

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \overline{\mathbf{x}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}?$$

Compute the optimal choice for the relaxation parameter ω in the JOR method. Compare the number of iterations needed to achieve an error of 10^{-6} in maximum norm with the optimal ω parameter and with $\omega = 0.01$ (we start the iteration from the zero vector).

Householder reflection, Givens rotation, QR decomposition - `qr`, `mldivide (\)`

PROBLEM. 38. Give a Householder reflection matrix to the vector $\overline{\mathbf{x}} = [2, 6, -3]^T$. Give a Givens rotation matrix to the 2:3 subvector of $\overline{\mathbf{x}}$.

PROBLEM. 39. Give the QR decomposition of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 1 & 3 \\ 0 & 2 \end{bmatrix}$$

with Givens rotations. Compare the result with the result of the command `qr(A)`.

PROBLEM. 40. Show that if $\mathbf{A} = \mathbf{QR}$ is a QR decomposition of the non-singular matrix $\mathbf{A}$ such that all diagonal elements of $\mathbf{R}$ are positive, then the decomposition is unique.

PROBLEM. 41. Solve the over-determined system

$$\begin{aligned} 0x + 0y &= 3 \\ x + 3y &= 4 \\ 2y &= 1 \end{aligned}$$

with the `mldivide` command, with the solution of the normal equation, and with QR decomposition.

PROBLEM. 42. Give the second degree polynomial for which the graph of the polynomial is closest (in the sense of least squares: $\sum_i (ax_i^2 + bx_i + c_i - y_i)^2 \to \min$) to the points (1,1), (0,0), (1,3), (2,2).

Eigenvalue problems - `eig, eigs, hess`

PROBLEM. 43. By the use of Gershgorin's theorem give estimations for the eigenvalues of the matrix

$$\mathbf{A} = \begin{bmatrix} 4.2 & -0.1 & 0.2 & 0.1 \\ 0.05 & 3 & -0.1 & -0.05 \\ 0.5 & -0.5 & 2 & 0.1 \\ 0.1 & 0.2 & 0.3 & -1 \end{bmatrix}.$$

Give an estimation for the spectral radius of the matrix.

PROBLEM. 44. By the use of the Bauer–Fike theorem estimate the change of the eigenvalues of the matrices in the case when we add 0.1 to the element $a_{2,1}$.

$$a)\ \mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad b)\ \mathbf{A} = \begin{bmatrix} 9 & 2 \\ 1 & 3 \end{bmatrix}$$

PROBLEM. 45. Compute the strictly (or single) dominant eigenvalue and the eigenvector of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

using the power method with the starting vector $\overline{\mathbf{x}}_0 = [6, 7, 3]^T$. Compute the eigenvalue (and the eigenvector) closest to 1 starting from the vector $[3, -3, -5]^T$. Solve the problems manually (two iteration steps are enough) and by the use of the provided Matlab code. What is the second largest eigenvalue in absolute value?

PROBLEM. 46. Let the matrix $\mathbf{A}$ be defined by the Matlab command `toeplitz([20:-1:1])`. Approximate all the eigenvalues of the matrix with the QR iteration. Give the error of the approximations after 100 steps.

PROBLEM. 47. Compute the eigenvalues of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

manually using the QR iteration. How can we make the algorithm usable?

PROBLEM. 48. Reduce (manually) matrix $\mathbf{A}$ to an upper Hessenberg form such that its eigenvalues do not change. Apply the QR iteration with the new reduced matrix.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 1 & 2 \\ 4 & 2 & 1 \end{bmatrix}$$

Solution of nonlinear equations - `fzero, fsolve, fminunc`

PROBLEM. 49. In 1225, Fibonacci gave the only real zero of the polynomial $p(x) = x^3 + 2x^2 + 10x - 20$ to 9 decimal places as $x^\star = 1.368808107$. The exact value is

$$x^\star = \frac{1}{3}\left(\sqrt{27}\sqrt{5240} + 352\right)^{1/3} - \frac{26/3}{\left(\sqrt{27}\sqrt{5240} + 352\right)^{1/3}} - \frac{2}{3}.$$

Show that p has exactly one zero in the interval [0,1]. Approximate the zero with the

a) bisection method (only a few steps, how many step we need to achieve an error of 10^{-6}?),

b) Newton's method (choose an appropriate starting value, give an error estimate after the fourth step, observe the order of the convergence),

c) fixed point iteration using the fixed point reformulation

$$x = \frac{20}{x^2 + 2x + 10}$$

(check the assumptions of the Banach fixed point theorem, observe the order of the convergence).

d) with the `fsolve` command of Matlab.

PROBLEM. 50. Try to solve the equation $p(x) := x^3 - 5x^2 + 8x - 4 = 0$ with the Newton's method starting from the point $x_0 = 3$. The method will find the root $x^\star = 2$. Determine the convergence order of the sequence obtained. How could we get back the second order convergence of the Newton's method?

PROBLEM. 51. We would like to compute the zero nearest to 1 of the function

$$f(x) = \frac{x}{8}(63x^4 - 70x^2 + 15).$$

By the use of Matlab, let us show that the iteration formula

$$x_{k+1} = x_k - f(x_k)/10 =: F(x_k)$$

can be used to obtain the zero of the function on the interval $[0.8, 1]$ (check the assumptions of the Banach fixed point theorem). Execute the iteration with $x_0 = 1$ and compute the zero in question of the function to 6 correct decimal places.

PROBLEM. 52. We compute the root $x^\star = 2$ of the equation

$$\frac{2x^2 - 3x - 2}{x - 1} = 0$$

using the iteration

$$x_{k+1} = x_k - 1 + \frac{1}{x_k - 1}.$$

What is the order of the convergence? Where can we start the iteration?

PROBLEM. 53. Find a solution of the system

$$\begin{aligned} x^2 + 2y^2 - y - 2z &= 0 \\ x^2 - 8y^2 + 10z &= 0 \\ x^2 - 7yz &= 0 \end{aligned}$$

near the point $(1, 1, 1)$. Use Newton's method and Matlab's built-in solver.

PROBLEM. 54. Find a local minimizer of the function $f(x_1, x_2) = (1-x_1)^2 + 5(x_2 - x_1^2)^2$ around the point (2,2). Use Matlab's `fminunc` function.

Interpolation with polynomials - `polyfit, polyval, spline, ppval`

PROBLEM. 55. Construct the interpolation polynomial to the points (-1,6), (0,3), (1,2) with Lagrange's and Newton's methods.

PROBLEM. 56. We interpolate the function $f(x) = \ln(x + 1)$ on the nodes 0, 0.6, 0.9. Give an upper bound for the interpolation error at the point $x = 0.45$.

PROBLEM. 57. We would like to interpolate the function $f(x) = \sin x$ with a piecewise linear function on the interval $[0, \pi]$ using equidistant nodes. Give an upper estimate to the step size that guarantees an interpolation error less than 0.001 on the whole interval.

PROBLEM. 58. We interpolate Runge's function $f(x) = 1/(1 + x^2)$ in the interval $[-1, 1]$ on 12 Chebyshev nodes. Estimate the interpolation error (use Matlab to compute and estimate the derivatives of the function).

PROBLEM. 59. Give the polynomial p with the smallest degree possible such that $p(1) = 2$, $p(3) = 1$, $p'(1) = 1$ and $p'(3) = 2$.

PROBLEM. 60. Construct the piecewise cubic natural spline that interpolates the points (1,2), (2,1), (3,1).

Trigonometric interpolation - `fft, ifft`

PROBLEM. 61. Let us construct the trigonometric interpolation polynomial to the function $f(x) = |x - \pi|$ on 4 equidistant interpolation nodes.

PROBLEM. 62. The values of a function f at 6 equidistant nodes on the interval $[0, 2\pi]$ are $0, 1, 1, 0, -1, -1$, respectively. a) Calculate the coefficient of $\sin x$ in the trigonometric interpolation polynomial manually. b) Use Matlab's `fft` command to calculate the complex and real discrete Fourier coefficients. c) Apply the fast Fourier transform manually to calculate the complex and real discrete Fourier coefficients.

PROBLEM. 63. The vector f in the code found in the m-file represents a sampling vector from a noisy 2π periodic signal. Use the fast Fourier transform to filter out the noise from the signal.

Numerical differentiation

PROBLEM. 64. Calculate the approximate first derivative of the function $f(x) = 1/x$ at the point $x_0 = 0.05$ using the forward difference formula with mesh sizes $h_1 = 0.0016$ and $h_2 = 0.0008$. Use Richardson extrapolation to give a better estimate for the derivative.

PROBLEM. 65. Show that the forward finite difference formula

$$D = \frac{-3f_0 + 4f_1 - f_2}{2h}$$

approximates the first order derivative with convergence order 2. Check the result on the function of the previous problem. Use Richardson extrapolation to give a better estimate for the derivative.

Numerical integration - `integral`

PROBLEM. 66. Calculate the Newton–Cotes coefficients $N_c^{4,k}$ and give an approximate value to the integral

$$\int_0^{\pi} \sin x \, dx$$

with them. Use Matlab's `integral` command to calculate the integral.

PROBLEM. 67. Let us apply the composite Simpson's rule to estimate the integral of the previous problem. Let us verify the order of the convergence of the method using Matlab. How small should the length of the subintervals be to achieve an error less than 10^{-6}?

PROBLEM. 68. Let us construct the four-point Gauss-Chebyshev formula on the interval $[-1, 1]$! What is the order of the exactness of the method? Verify the result with Matlab. (The nodes are the zeros of the 4th degree Chebyshev polynomial and all weights are $\pi/4$.)

PROBLEM. 69. Let us apply the three-point Gauss–Legendre formula in a composite way to estimate the integral $\int_0^1 e^{-x^2} \, dx$. What is the order of the convergence? Check it! (The third degree Legendre polynomial is $(5x^3 - 3x)/2$ and the weights are 5/9, 8/9, 5/9, respectively.)

Solution of initial value problems - `ode45, ode23s`

PROBLEM. 70. Solve the initial value problem $y' = \text{arctg} y, y(0) = 1$ with the explicit Euler method on the interval $[0, 1]$. Give an estimate to the step size to guarantee an error below 10^{-4}. (The error formula is

$$\|\bar{\mathbf{e}}_k\| \leq e^{(x_{\max} - x_0)L} h(x_{\max} - x_0) M_2/2 \quad .)$$

PROBLEM. 71. Solve the initial value problem $y' = x^2(1 + y), y(0) = 3$ with the RK4 method. Verify the order of the convergence of the method. Repeat the investigation with the Heun method.

PROBLEM. 72. Consider the following problem of reaction kinetics on the interval [0,50], where we use the initial values 1,1,0, respectively. Solve the problem with Matlab's `ode45`, `ode23s` functions and with the implicit Euler method. Compare the results.

$$\begin{aligned} c_1' &= -0.013c_1 - 1000c_1c_3 \\ c_2' &= -2500c_2c_3 \\ c_3' &= -0.013c_1 - 1000c_1c_3 - 2500c_2c_3 \end{aligned}$$

Problem. 73. Construct the two-step BDF formula (BDF2). Investigate its stability, consistency and convergence. Verify its order of convergence on the initial value problem $y' = x^2(1+y), y(0) = 3$ ($x \in [0,1]$). Calculate the first step using the RK2 method.

Problem. 74. Construct the two-step Adams–Bashforth formula and perform similar analysis to that of the previous problem.

Solution of boundary value problems

Problem. 75. Using the shooting method, solve the boundary value problem $y'' = y + 4e^x, y(0) = 1, y(1/2) = 2e^{1/2}$! Solve the initial value problems with the Heun method, and apply the bisection method to solve the nonlinear equations. (Exact solution: $y(x) = (2x+1)e^x$.)

Problem. 76. Solve the boundary value problem in the previous problem using the finite difference method.