Volume of qubit channels and its distribution over classical channels

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1 Introduction

2 Preliminary lemmas

3 The main result
- Volume of general qubit channels
- Volume of unital qubit channel

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MOTIVATION

	Kolmogorovian probability	Quantum probability
States	1. Distributions on a finite set $\mathcal{S}$, $\|\mathcal{S}\| = n$	2. Density matrices in $\mathbb{C}^{n\times n}$
Transitions	3. Markov matrices $\supseteq$ doubly stochastic matrices	4. Quantum channels $\supseteq$ unital channels

1. $\Delta_{n-1} = \left\{ p \in [0,1]^n : \sum_{i=1}^{n} p_i = 1 \right\}$ the $(n-1)$-simplex.
2. $\mathcal{M}_n^{\mathbb{C}} = \{ D \in \mathbb{C}^{n\times n} : D \geq 0, \operatorname{Tr} D = 1 \}$
3. (A) $\mathcal{S}_1 \to \mathcal{S}_2$: $(\Delta_{|\mathcal{S}_2|-1})^{\mathcal{S}_1}$
   (B) $\mathcal{S} \to \mathcal{S}$: B_n (Birkhoff-polytope)
4. (A) $\mathcal{M}_n^{\mathbb{C}} \to \mathcal{M}_m^{\mathbb{C}}$ CPT maps.
   (B) Identity-preserving $\mathcal{M}_n^{\mathbb{C}} \to \mathcal{M}_m^{\mathbb{C}}$ CPT maps.

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Historical background

$V(B_n)$

Explicit value of $V(B_n)$ if known for $n \leq 10$. Assyptotic formula for $V(B_n)$:

$$V(B_n) = \exp\left(-(n-1)^2 \log(n) + n^2 - \left(n - \frac{1}{2}\right)\log(2\pi) + \frac{1}{3} + o(1)\right)$$

(R. Canfield and B. McKay, 2007).

$V(\mathcal{M}_n^{\mathbb{C}})$

$$V(\mathcal{M}_n^{\mathbb{C}}) = \frac{\pi^{\binom{n}{k}}}{(n^2-1)!}\prod_{i=1}^{n-1} i!$$

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Quantum channels

Definition

A map $Q : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$ is *completely positive* (CP) if $\forall k$ $\mathrm{id}_k \otimes Q : \mathbb{C}^{k\times k} \otimes \mathbb{C}^{n\times n} \to \mathbb{C}^{k\times k} \otimes \mathbb{C}^{m\times m}$ is positive.

Theorem (Choi representation, Choi matrix)

The followings are equivalent:

I. $Q : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$ *is CP.*

II. $Q : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$ *is n-positive.*

III. *The matrix* $(id_n \otimes Q)\left(\sum_{i,j=1}^{n} E_{ij} \otimes E_{ij}\right)$ *is positive.*

The undelying classical channel

Let $Q : \mathcal{M}_n^{\mathbb{C}} \to \mathcal{M}_m^{\mathbb{C}}$ a CPT map, ι_n and π_m are the canonical inclusion and projection, respectively.

$$\begin{array}{ccc} \mathcal{M}_n^{\mathbb{C}} & \xrightarrow{Q} & \mathcal{M}_m^{\mathbb{C}} \\ \iota_n \uparrow & & \downarrow \pi_m \\ \Delta_{n-1} & \xrightarrow{\exists ! P_Q} & \Delta_{m-1} \end{array}$$

Qubit case:

$$\mathcal{M}_2^{\mathbb{C}} \ni \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto aQ_{11} + bQ_{12} + cQ_{21} + dQ_{22}.$$

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \Rightarrow P_Q = \begin{bmatrix} \mathrm{diag}(Q_{11}) \\ \mathrm{diag}(Q_{22}) \end{bmatrix}$$

The manifold of qubit channels

Choi representation:

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \Longrightarrow Q_{11}, Q_{22} \in \mathcal{M}_2, Q_{21} = Q_{12}^*, \operatorname{Tr} Q_{12} = 0$$

General qubit channels:

$$\mathcal{Q}_{\mathbb{R}} = \{Q \in \mathbb{R}^{4\times 4} | Q : \mathcal{M}_2^{\mathbb{R}} \to \mathcal{M}_2^{\mathbb{R}}, Q > 0\} \subset \mathbb{R}^7$$
$$\mathcal{Q}_{\mathbb{C}} = \{Q \in \mathbb{C}^{4\times 4} | Q : \mathcal{M}_2^{\mathbb{C}} \to \mathcal{M}_2^{\mathbb{C}}, Q > 0\} \subset \mathbb{R}^{12}$$

Unital qubit channels:

$$Q_{11} + Q_{22} = I$$

$$\mathcal{Q}_{\mathbb{R}}^1 = \{Q \in \mathbb{R}^{4\times 4} | Q : \mathcal{M}_2^{\mathbb{R}} \to \mathcal{M}_2^{\mathbb{R}}, Q > 0, Q(I) = I\} \subset \mathbb{R}^5$$
$$\mathcal{Q}_{\mathbb{C}}^1 = \{Q \in \mathbb{C}^{4\times 4} | Q : \mathcal{M}_2^{\mathbb{C}} \to \mathcal{M}_2^{\mathbb{C}}, Q > 0, Q(I) = I\} \subset \mathbb{R}^9$$

Some facts from linear algebra

Proposition

$A > 0 \Leftrightarrow (\forall U)\, U^* A U > 0$, *where U is unitary.*

Proposition

1. $A > 0 \Leftrightarrow (\forall i)\ \det(A_i) > 0$.
2. *If* $A_n = \left[\begin{array}{c|c} A_{n-1} & x_n \\ \hline x_n^* & a_{nn} \end{array}\right] > 0$, *then*

$$\det(A_n) = a_{nn} \det(A_{n-1})\left(1 - \left\langle x_n, (A_{n-1})^{-1} x_n \right\rangle\right).$$

Abbreviations

A simple consequence of the beta integral:

$$G_{a,b} := \int_0^1 x^a (1-x^2)^b \,\mathrm{d}x = \frac{1}{2} \frac{\Gamma(b+1)\Gamma\left(\frac{a+1}{2}\right)}{\Gamma\left(\frac{a}{2}+b+\frac{3}{2}\right)}$$

The surface F_{n-1} of the unit sphere in an n dimensional space:

$$F_{n-1} = \frac{n\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$$

Our main tool

Lemma

Assume that $T \in \mathbb{K}^{n\times n}$, $T > 0$, $l \in \mathbb{R}$ *and* $\mu > 0$. *Let* L *be an* m*-dimensional subspace and* $x \in \mathbb{K}^n$ *fixed. Set*

$$\mathrm{E}^{\mathbb{K}}(T,\mu,L,x) = \{y \in L |\ \langle x+y, T(x+y)\rangle < \mu\}\,, \quad T_{ij} \in \mathbb{K}$$

then

$$\int_{\mathrm{E}^{\mathbb{K}}(T,\mu,L,x)} (\mu - \langle x+y, T(x+y)\rangle)^l \,\mathrm{d}\lambda_{dm}(y) = \frac{F_{dm-1}G_{dm-1,l}}{\det(T|_L)^{d/2}}(\mu - ||z_0||^2)_+^{\frac{dm}{2}+l}$$

where $M = T(L)$, $z_0 = P_{M^\perp}\sqrt{T}x$ *and* $d = \dim_{\mathbb{R}}(\mathbb{K})$.

The sketch of the proof (real case)

Substitution:

$$z(y_1, \ldots, y_m) = \frac{1}{\sqrt{\mu}} \sqrt{T} \left(x + \sum_{i=1}^{m} y_i e_i \right)$$

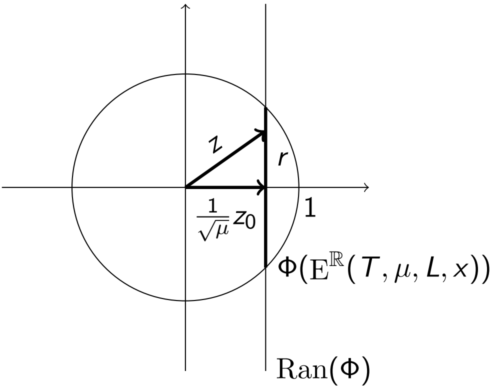
Jacobian: $\dfrac{\mu^{\frac{m}{2}}}{\sqrt{\det(T|_L)}}$

$$\int_{\mathrm{E}^{\mathbb{R}}(T,\mu,L,x)} (\mu - \langle x+y, T(x+y)\rangle)^l \,\mathrm{d}\lambda_m(y) =$$

$$= \frac{\mu^{\frac{m}{2}+l}}{\sqrt{\det(T|_L)}} \int_{\Phi(\mathrm{E}^{\mathbb{R}}(T,\mu,L,x))} (1 - ||z||^2)^l \,\mathrm{d}\lambda_m(z).$$

and then spehrical coordinates on $\Phi(\mathrm{E}^{\mathbb{R}}(T, \mu, L, x))$.

Region of integration:



where $z_0 = P_{M^\perp} \sqrt{T} x$

Theorem

The volume of the space of qubit channels and the distribution of volume over classical channels can be expressed as follows.

Real channels:

$$V(\mathcal{Q}_{\mathbb{R}}) = \frac{4\pi^3}{105},$$

$$V(a,f) = \frac{2^6}{15} F_1^2 G_{1,0} G_{1,\frac{1}{2}} \times \begin{cases} (af)^{3/2}\left(5(1-a)(1-f) - af\right) & \text{if } a+f<1 \\ ((1-a)(1-f))^{3/2}(5af - (1-a)(1-f)) & \text{if } a+f \geq 1. \end{cases}$$

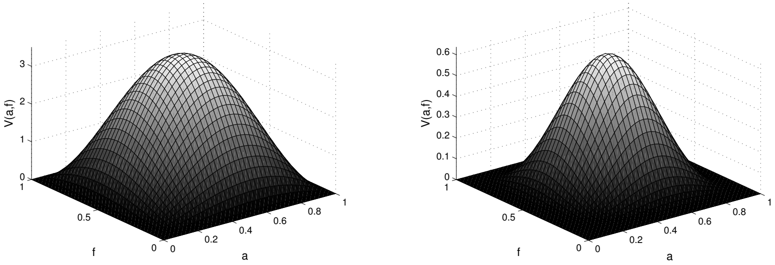
Complex channels:

$$V(\mathcal{Q}_{\mathbb{C}}) = \frac{2\pi^5}{4725},$$

$$V(a,f) = \frac{2^7}{60} F_1 F_3^2 G_{3,0} G_{3,1} \times \begin{cases} a^3 f^3 [10((1-a)(1-f) - af)^2 + \\ 15af(1-a)(1-f) - 9a^2f^2] & \text{if } a+f<1 \\ (1-a)^3(1-f)^3[10((1-a)(1-f) - af)^2 + \\ 15af(1-a)(1-f) - 9(1-a)^2(1-f)^2] & \text{if } a+f \geq 1. \end{cases}$$

Distribution of volume over classical channels

Underlying classical channel: $\begin{bmatrix} a & 1-a \\ 1-f & f \end{bmatrix}$



(A) $\mathcal{Q}_{\mathbb{R}}$. (B) $\mathcal{Q}_{\mathbb{C}}$.

Figure : Graph of $V(a,f)$.

Parametrization of $\mathcal{Q}_{\mathbb{R}}$ and $\mathcal{Q}_{\mathbb{C}}$:

$$Q = \begin{bmatrix} a & b & c & d \\ \bar{b} & 1-a & e & -c \\ \bar{c} & \bar{e} & f & g \\ \bar{d} & -\bar{c} & \bar{g} & 1-f \end{bmatrix} \overset{\text{unitary transform}}{\Longrightarrow} A = \begin{bmatrix} a & c & b & d \\ \bar{c} & f & e & g \\ \bar{b} & \bar{e} & 1-a & -c \\ \bar{d} & \bar{g} & -\bar{c} & 1-f \end{bmatrix}$$

The corresponding volume form:

$\mathcal{Q}_{\mathbb{R}}$	$2^4 \, \mathrm{d}\lambda_7$
$\mathcal{Q}_{\mathbb{C}}$	$2^7 \, \mathrm{d}\lambda_{12}$

Lemma

Let $\mathbb{K}^{n\times n} \ni A > 0$, $T = \det(A)A^{-1}$, $L \leq \mathbb{K}^n$ *a subspace,* $x \in L^\perp$ *and* $M = \sqrt{T}L$. *If* $\dim(L^\perp) = 1$, *then*

$$||P_{M^\perp}\sqrt{T}x||^2 = \frac{\det(A)}{\langle x, Ax\rangle}||x||^4.$$

The critical step of the calculation ($\mathcal{Q}_{\mathbb{C}}$)

$$A = \left[\begin{array}{ccc|c} a & c & b & d \\ \bar{c} & f & e & g \\ \bar{b} & \bar{e} & 1-a & -c \\ \hline \bar{d} & \bar{g} & -\bar{c} & 1-f \end{array}\right]$$

Blue: fixed
Red: running (L_3)

Positivity condition: $(1-f)\det(A_3)(1-\langle x_3, (A_3)^{-1} x_3\rangle) \geq 0.$

$$V(A_3) = \int\limits_{\mathrm{E}^{\mathbb{C}}(\det(A_3)(A_3)^{-1}, (1-f)\det(A_3), L_3, x_3)} 2^7 \, \mathrm{d}\lambda_4$$

$$= \frac{2^7 F_3 G_{3,0} \left((1-f) - \frac{|c|^2}{1-a}\right)_+^2 \det(A_3)^2}{\det(T_3|_{L_3})}$$

$$= \frac{2^7 F_3 G_{3,0}}{(1-a)^3} \left((1-a)(1-f) - |c|^2\right)_+^2 \det(A_3)$$

Theorem

The volume of the space of unital qubit channels and the distribution of volume over classical channels can be expressed as follows.

Real unital channels:

$$V(\mathcal{Q}^1_{\mathbb{R}}) = \frac{4\pi^2}{15},$$

$$V(a) = 2^5 F_0 F_1 G_{0,\frac{1}{2}} G_{1,1} a^2 (1-a)^2.$$

Complex unital channels:

$$V(\mathcal{Q}^1_{\mathbb{C}}) = \frac{2\pi^4}{315},$$

$$V(a) = 2^2 \pi^4 a^4 (1-a)^4.$$

Distribution of volume over classical channels

Underlying classical channel: $\begin{bmatrix} a & 1-a \\ 1-a & a \end{bmatrix}$

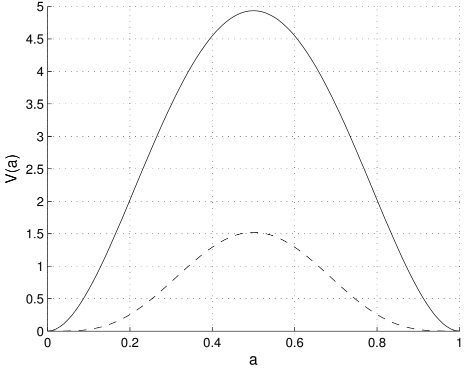


Figure : Graph of $V(a)$ for $\mathcal{Q}^1_{\mathbb{R}}$ (solid) and $\mathcal{Q}^1_{\mathbb{C}}$ (dashed).

Parametrization of $\mathcal{Q}^1_{\mathbb{R}}$ and $\mathcal{Q}^1_{\mathbb{C}}$:

$$Q = \begin{bmatrix} a & b & c & d \\ \bar{b} & 1-a & e & -c \\ \bar{c} & \bar{e} & 1-a & -b \\ \bar{d} & -\bar{c} & -\bar{b} & a \end{bmatrix} \overset{\text{unitary transform}}{\Longrightarrow} A = \begin{bmatrix} 1-a & e & b & -c \\ \bar{e} & 1-a & c & -b \\ \bar{b} & \bar{c} & a & d \\ -\bar{c} & -\bar{b} & \bar{d} & a \end{bmatrix}$$

The corresponding volume form:

$\mathcal{Q}^1_{\mathbb{R}}$	$2^4 \, \mathrm{d}\lambda_5$
$\mathcal{Q}^1_{\mathbb{C}}$	$2^7 \, \mathrm{d}\lambda_9$

Lemma

If $L_3 = Span\{(0,0,1)^T\}$ and $M = \sqrt{A_3^{-1}}(L_3)$, then

$$\sqrt{A_3^{-1}} P_{M^\perp} \sqrt{A_3^{-1}} = \begin{bmatrix} A_2^{-1} & \mathbf{0} \\ \mathbf{0}^* & 0 \end{bmatrix}.$$

The critical step of the calculation $(\mathcal{Q}_{\mathbb{C}})$

$$A = \begin{bmatrix} 1-a & e & b & -c \\ \bar{e} & 1-a & c & -b \\ \bar{b} & \bar{c} & a & d \\ -\bar{c} & -\bar{b} & \bar{d} & a \end{bmatrix}$$

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$$V(A_3) = \int_{\mathrm{E}^{\mathbb{C}}(\det(A_3)(A_3)^{-1}, a\det(A_3), L_3, x_3)} 2^7 \, \mathrm{d}\lambda_2 =$$

$$= \frac{2^6 F_1}{\det(A_2)} \left(a - \left\langle x_3, \begin{pmatrix} A_2^{-1} & \mathbf{0} \\ \mathbf{0}^* & 0 \end{pmatrix} x_3 \right\rangle \right)_+ \det(A_3)$$

Uniform sampling algorithm ($\mathcal{Q}_{\mathbb{R}}$)

Step 1: Generate $a, f \sim \mathcal{U}([0,1])$ independently.

Step 2: Generate $x_1 \sim \mathcal{U}(-\sqrt{af}, \sqrt{af})$ and set $A_2 = \begin{bmatrix} a & x_1 \\ x_1^* & f \end{bmatrix}$.

Step 3: Generate $y_2 \sim \mathcal{U}(\{r \in \mathbb{R}^2 : ||r|| \leq \sqrt{f}\})$ and set $A_3 = \begin{bmatrix} A_2 & x_2 \\ x_2^* & 1-a \end{bmatrix}$, where $x_2 = \sqrt{A_2} y_2$.

Step 4: Compute the projection P onto the subspace $\mathrm{Span}(\{\sqrt{A_3}\, e_3\})$ and set $z = -x_1(1-f)^{-1/2} P \sqrt{A_3^{-1}}\, e_3$.

Step 5: **If** $||z|| > 1$, **then goto** Step 2.

Step 6: Generate $y_3 \sim \mathcal{U}(\{r \in \mathbb{R}^2 : ||r|| \leq \sqrt{1-||z||^2}\})$ and set $A = \begin{bmatrix} A_3 & x_3 \\ x_3^* & 1-f \end{bmatrix}$, where $x_3 = \sqrt{1-f}\sqrt{A_3}([e_1, e_2] y_3 + z)$.

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The trace-distance contraction coefficient

	Kolmogorovian probability	Quantum probability
Distance of states	$\lvert\mathbb{P}-\mathbb{Q}\rvert_1=\sum_{s\in\mathcal{S}}\lvert P(s)-Q(s)\rvert$	$\lvert\lvert\rho-\sigma\rvert\rvert_1=\mathrm{Tr}\,\lvert\rho-\sigma\rvert$
Contractivity of transitions	Dobrushin ergodic coefficient	$\eta^{\mathrm{Tr}}(Q)$

$$\eta^{\mathrm{Tr}}(Q)=\sup\left\{\frac{\lvert\lvert Q(\rho)-Q(\sigma)\rvert\rvert_1}{\lvert\lvert\rho-\sigma\rvert\rvert_1}:\rho,\sigma\in\mathcal{M}_2\right\}$$

Questions

1. What is the supremum of η^{Tr} over a fixed classical channel?
2. What is the infimum of η^{Tr} over a fixed classical channel?
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Computation of η^{Tr}

The qubit channel $Q : \mathcal{M}_2 \to \mathcal{M}_2$ in the Pauli bases

$$Q\left(\frac{1}{2}(I + x\cdot\sigma)\right) = \frac{1}{2}\left(I + (v + Tx)\cdot\sigma\right),$$

where $v \in \mathbb{R}^3$ and $T \in \mathbb{R}^{3\times 3}$.

$$\eta^{\mathrm{Tr}}(Q) = ||T||_\infty$$

$$Q = \begin{bmatrix} a & b & c & d \\ \bar{b} & 1-a & e & -c \\ \bar{c} & \bar{e} & f & g \\ \bar{d} & -\bar{c} & \bar{g} & 1-f \end{bmatrix} \Longrightarrow T = \begin{bmatrix} \Re(d+e) & \Im(d+e) & \Re(b-g) \\ -\Im(d-e) & \Re(d-e) & -\Im(b-g) \\ 2\Re(c) & 2\Im(c) & a-f \end{bmatrix}$$

Theorem (Answer to Question 1.)

$(\forall a, f \in [0,1])$, $\forall x \in \left(|a-f|, \sqrt{(1-a)f} + \sqrt{a(1-f)}\right)$

$$\exists Q \in \mathcal{Q}_{\mathbb{R}}(a,f) \subset \mathcal{Q}_{\mathbb{C}}(a,f) \text{ such that } \eta^{\mathrm{Tr}}(Q) = x.$$

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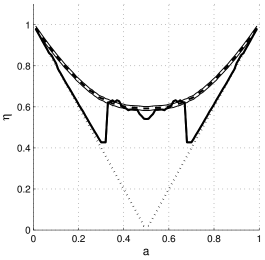
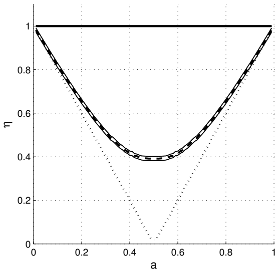
Distribution η^{Tr} over classical channels

Conjecture (Answer to Question 2.)

$$\inf\{\eta^{\mathrm{Tr}}(Q) : Q \in \mathcal{Q}_{\mathbb{C}}(a, f)\} = |a - f|$$



(A) $\mathcal{Q}^1_{\mathbb{R}}$



(B) $\mathcal{Q}^1_{\mathbb{C}}$

Figure : Minimal value (dotted) of η^{Tr}, mode of η^{Tr} (thick), expectation of η^{Tr} (dashed) and confidence band (solid) corresponding to the expectation ($n = 1000$, $\alpha = 5 \times 10^{-5}$).

Volume of quantum channels

Partition of the Choi matrix:

$$Q_n = \left[\begin{array}{c|c} Q_{n-1} & C_n \\ \hline C_n^* & D_n \end{array}\right] \in \mathbb{C}^{nm\times nm},$$

where $D_n \in \mathcal{M}_m^{\mathbb{C}}$ and $C_n = [C_{n,1},\dots,C_{n,n-1}]^T$, $C_{n,j} \in \mathbb{C}^{m\times m}$, $\mathrm{Tr}\, C_{n,j} = 0$.

If Q_{n-1} D_n is fixed, then C_n must satisfy one of the following inequalities:

$$\mathrm{id}_{\mathbb{C}^{(n-1)m}} > \left(Q_{n-1}^{-1/2} C_n D_n^{-1/2}\right)\left(Q_{n-1}^{-1/2} C_n D_n^{-1/2}\right)^*$$

or

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