1. Tétel [Wiener] $\mu \in M(\mathbb{T})$ (véges Borel-mérték), $\mathbb{T} = \{z \mid |z| = 1\} \subseteq \mathbb{C}$

$$\hat{\mu}(n) = \int_{\mathbb{T}} z^{-n} \, d\mu(z) \quad \text{Fourier-együttható}$$

$$\Rightarrow \quad \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N+k}^{N+k} |\hat{\mu}(n)|^2 = \sum_{\lambda \in \mathbb{T}} |\mu(\{\lambda\})|^2 \qquad k \in \mathbb{Z}\text{-re egyenletesen}$$

2. Tétel: H Hilbert tér, $T \in \mathcal{L}(H)$, $\|T\| \leq 1$

$P_\lambda : H \to H$ ortogonális projekció $\ker(\lambda - T)$-re. $(\lambda \in \mathbb{C})$

$$\Rightarrow \quad \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |(T^n x \mid T^{kx} y)|^2 = \sum_{\lambda \in \mathbb{T}} |(P_\lambda x \mid P_\lambda y)|^2 = \sum_{\lambda \in \mathbb{T}} |(P_\lambda x \mid y)|^2$$

$k \in \mathbb{N}$-re egyenletesen.

3 Lemma: T kontrakció

(a) $\lambda \in \Pi$, $Tx = \lambda x \Rightarrow T^* x = \bar{\lambda} x$

(b) $\lambda, \mu \in \Pi$, $\lambda \neq \mu$ $Tx = \lambda x$ és $Ty = \mu y \Rightarrow x \perp y$

(c) $\lambda \in \Pi \Rightarrow T^* P_\lambda = P_\lambda T^*$

Biz: (a) $\|T^* x - \bar{\lambda} x\|^2 = \|T^* x\|^2 + \|x\|^2 - (T^* x \mid \bar{\lambda} x) - (\bar{\lambda} x \mid T^* x)$

$\leq \|x\|^2 + \|x\|^2 - \|x\|^2 - \|x\|^2 = 0$

(b) $(\lambda - \mu)(x \mid y) = (Tx \mid y) - (x \mid T^* y) = 0$

(c) $(P_\lambda T^* x \mid y) = (T^* x \mid P_\lambda y) = (x \mid T P_\lambda y) = \bar{\lambda}(x \mid P_\lambda y) = \bar{\lambda}(x \mid P_\lambda y) = (P_\lambda T^* x \mid y)$ ∎

4 Tétel: [Glicksberg - de Leeuw - Jacobs]: T kontrakció

$\Rightarrow \quad H = H_s \oplus H_r$ ortogonális felbontás

$H_s = \overline{\lim} \{ x \mid \exists\, \lambda \in \Pi \,,\, Tx = \lambda x \}$

$H_r = \{ x \mid T^{n_k} x \to 0$ gyengén valamely (n_k) részsorozattal$\}$

$T|_{H_s}$ unitér, pályái erősen kompaktak

$x \in H_r \iff \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |(T^n x \mid y)| = 0 \quad \forall\, y \in H$

Megj: $P: H \to H$ ortogonális projekció H_s-re

$P = \sum_{\lambda \in \Pi} P_\lambda$ ortogonális összeg (Lemma 3), erősen konvergens

2 Tétel: [Koopman, von Neumann] $(c_n) \subseteq \mathbb{C}$ korlátos. Ekv:

(i) $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_n = 0$ (ii) $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_n^2 = 0$ (iii) $c_n \to 0$ sztochasztikusan

6. Lemma [Jones-Lin] $x \in H_r \iff$

$$\sup_{\|y\| \le 1} \frac{1}{N} \sum_{n=1}^{N} |(T^n x | y)| \to 0 \quad (N \to \infty)$$

7 Biz:

$$\frac{1}{N} \left| \sum_{n=1}^{N} |(T^n x | y)|^2 - d \right| \le \frac{1}{N} \sum_{n=1}^{N} \left| |(T^n x | y)|^2 - |(T^n P_x | P_y)|^2 \right| + \frac{1}{N} \left| \sum_{n=1}^{N} |(T^n P_x | P_y)|^2 - d \right|$$

$$\le \frac{1}{N} \sum_{n=1}^{N} \underbrace{\left| (T^n x | y) - (T^n P_x | P_y) \right|}_{(T^n (x - P_x) | (y - P_y))} \left(|(T^n x | y)| + |(T^n P_x | P_y)| \right) + \frac{1}{N} \left| \sum_{n=1}^{N} |(T^n P_x | P_y)|^2 - d \right|$$

$x - P_x \in H_r$, $y - P_y \in H_r$

$\to 0 \quad (N \to \infty)$ egyenletesen $\|y\| \le 1$ esetén

$$\left|\left(T^n P_x \mid P_y\right)\right|^2 = \left|\left(\sum_\lambda \lambda^n P_\lambda x \mid \sum_\mu P_\mu y\right)\right|^2 = \left|\sum_\lambda \lambda^n \left(P_\lambda x \mid P_\lambda y\right)\right|^2 =$$

$$= \sum_\lambda \lambda^n \left(P_\lambda x \mid P_\lambda y\right) \cdot \sum_\mu \mu^{-n} \left(P_\mu y \mid P_\mu x\right)$$

$$\frac{1}{N} \sum_{n=1}^{N} \left|\left(T^n P_x \mid P_y\right)\right|^2 = \frac{1}{N} \sum_{n=1}^{N} \sum_{\mu,\nu} (\lambda\bar{\mu})^n \left(P_\lambda x \mid P_\lambda y\right)\left(P_\mu y \mid P_\mu x\right) = \sum_\lambda \left|\left(P_\lambda x \mid P_\lambda y\right)\right|^2 +$$

$$+ \sum_{\mu,\lambda} \frac{1}{N} \lambda\bar{\mu} \, \frac{|\lambda\bar{\mu}|^N - 1}{\lambda\bar{\mu} - 1} \left(P_\lambda x \mid P_\lambda y\right)\left(P_\mu y \mid P_\mu x\right) \to 0$$

Specialisan $y \rightsquigarrow T^{*k} y$ $\quad k \in \mathbb{N}$

$$\frac{1}{N} \sum_{n=1}^{N} \left|\left(T^n P_x \mid P T^{*k} y\right)\right|^2 = \sum_\lambda \underbrace{\left|\left(P_\lambda x \mid P_\lambda T^{*k} y\right)\right|^2}_{=1} + \sum_{\mu,\lambda} \frac{1}{N} \frac{(\lambda\bar{\mu})^{N+1} - 1}{\lambda\bar{\mu} - 1} \lambda^k \bar{\mu}^k \left(P_\lambda x \mid P_\lambda y\right)\left(P_\mu y \mid P_\mu x\right)$$

$\to 0$ korábban egyszerűsítés.

8. Példa: $\mu \in M(\mathbb{T})$ $L^2(\mathbb{T}, |\mu|)$, $Tf(z) = \bar{z} f(z)$, T unitér

$\mu = h|\mu|$ (Radon-Nikodym, $f \in L^\infty(\mathbb{T}, |\mu|)$

$$\hat{\mu}(n+k) = \int_{\mathbb{T}} z^{-(n+k)} \, d\mu = \int_{\mathbb{T}} z^{-(n+k)} h(z) \, |\mu|(z) = (T^n 1 \mid T^{*k} h)$$

$$P_\lambda f = 1_{[\lambda]} f \qquad (P_\lambda h \mid P_\lambda 1) = \int 1_{[\lambda]} h \, 1 \, d|\mu| < \mu(\{\lambda\})$$

Így Wiener tétel következik az absztrakt verzióból

9 Példa: H Hilbert, $T \in \mathcal{L}(H)$ kontrakció

$H = H_{cnu} \oplus H_{uni}$ (Szőkefalvi-Nagy felbontás)

$T|_{H_{uni}}$ unitér $T|_{H_{cnu}}$ teljesen nem unitér

$\mu_{x,y}$ Spektráltételből $(T^n x \mid y) = \int_{\mathbb{T}} z^n \, d\mu_{x,y}$ $x, y \in H_{uni}$

$H_{ani} = H_c \oplus H_d \qquad H_c = \{ x \mid \mu_{x,x} \text{ folgtonos} \}, \quad H_d = \{ x \mid \mu_{x,x} \text{ diskrit} \}$

$H_v := H_{cnn} \oplus H_c \qquad H_s := H_c$

<u>Foras</u>: $x \in H_{cnn} \quad (T^n x \mid y) \to 0 \quad (n \to \infty)$

<u>Wiener</u>: $x \in H_c \quad \frac{1}{N} \sum_{n=1}^{N} \int_{\mathbb{T}} z^n \, d\mu_{x,x} = \frac{1}{N} \sum_{n=1}^{N} \hat{\mu}_{x,x}(n) \to 0$

<u>Halmos</u>: $H_d = \overline{\text{lin}} \{ x \mid \exists \lambda \in \mathbb{T}, \ Tx = \lambda x \}$

$\Longrightarrow$ Glicksberg-de Leeuw-Jacobs

10. Példa: Host-Kra-Gowers uniformity normák

(X, φ) ergodikus mértékelméleti dinamikai rendszer, T Koopman op. $Tf = f \circ \varphi$

$$\|f\|_{U^1} = \left| \int_X f \, d\mu \right|$$

$f \in L^\infty$

$$\|f\|_{U^{d+1}} = \limsup_{N \to \infty} \left(\frac{1}{N} \sum_{n=0}^{N} \|T^n f \cdot \bar{f}\|_{U^d}^{2^d} \right)^{1/2^{d+1}} \qquad \text{(lineáris, folytonos)}$$

$$\|f\|_{U^2}^4 = \limsup_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \left| \int_X T^m \bar{f} \cdot f \right|^2 = \sum_{\chi \in \hat{T}} \|P_\chi f\|^4 \qquad \text{"}L^4\text{-Norma"}$$

$$\operatorname{Ker} \|\cdot\|_{U^2}^{\perp} = H|_s \qquad \overline{\operatorname{Ker} \|\cdot\|_{U^2}} \oplus H|_s = H = L^2$$