Potential Theory and Quadratic Programming

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MOTIVATION

X: locally compact Hausdorff space k l.s.c. positive (symmetric) kernel. $K \subset X$, $\mathcal{M}_1(K)$: the probability measures on K. Frostman, Choquet, Fuglede, Ohtsuka, etc.: The Wiener energy:

$$w(K) := \inf_{\mu \in \mathcal{M}_1(K)} \int_{X \times X} k(x,y) d\mu(x) \times \mu(y).$$

Frostman, Fuglede, etc. "maximum principle" $\Rightarrow$

$$w(K) = q(K) := \inf_{\mu \in \mathcal{M}_1(K)} \sup_{y \in K} \int_X k(x,y) d\mu(x).$$

Fuglede, Farkas - Révész, etc : $K, L \subset X$

$$q(K,L) = \inf_{\mu \in \mathcal{M}_1(K)} \sup_{y \in L} \int_X k(x,y) d\mu(x).$$

Adams - Hedberg p-capacity:

$$C_p(K)^{-\frac{1}{p}} = \inf_{\mu \in \mathcal{M}_1(K)} \sup_{\{f d\nu : f \geq 0, \|f\|_{p,\nu} \leq 1\}} \int_{\mathbb{R}^n} \int_X k(x,y) d\mu(x) f(y) d\nu(y).$$

ENERGY WITH RESPECT TO SETS OF MEASURES

$\mathcal{M}(X) := \{\mu : \mu$ is a positive, regular Radon measure on X, $\mu(X) < \infty\}$.

DEFINITION.

Let $H \subset X$, $L \subset Y$ and $\mathcal{R}(H) \subset \mathcal{M}(H)$, $\mathcal{S}(L) \subset \mathcal{M}(L)$.

$$\overline{q}(\mathcal{R}(H), \mathcal{S}(L)) := \inf_{\mu \in \mathcal{R}(H)} \sup_{\nu \in \mathcal{S}(L)} E(\mu, \nu) = \inf_{\mu \in \mathcal{R}(H)} \sup_{\nu \in \mathcal{S}(L)} \int_Y \int_X k(x, y) d\mu(x) d\nu(y),$$

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THEOREM 1.

Let k be a positive, symmetric l.s.c. kernel on $X \times X$. Let $H, L \subset X$ two subsets of X, $\mathcal{R}(H)$, $\mathcal{S}(H)$ are convex subsets of $\mathcal{M}(H)$, $\tilde{\mathcal{R}}(L)$, $\tilde{\mathcal{S}}(L)$ are w^*-compact convex subsets of $\mathcal{M}(L)$. Moreover let us suppose that

$$\inf_{\nu \in \tilde{\mathcal{R}}(L)} \int_L U^\mu d\nu = \inf_{\nu \in \tilde{\mathcal{S}}(L)} \int_L U^\mu d\nu \ \forall \mu \in \mathcal{M}(H),$$

$$\sup_{\nu \in \mathcal{R}(H)} \int_H U^\mu d\nu = \sup_{\nu \in \mathcal{S}(H)} \int_H U^\mu d\nu \ \forall \mu \in \mathcal{M}(L).$$

Then

$$\underline{q}\left(\mathcal{R}(H), \tilde{\mathcal{S}}(L)\right) = q\left(\tilde{\mathcal{R}}(L), \mathcal{S}(H)\right).$$

Lemma. (Farkas, Révész/Adams, Hedberg)

Let A be a compact convex subset of the Hausdorff topological vector space U and B be a convex subset of the linear space V. Let $f : A \times B \to (-\infty, \infty]$ be l.s.c. on A for fixed $y \in B$, and assume that f is convex in the first and concave in the second variable. Then

$$\sup_{y\in B} \inf_{x\in A} f(x,y) = \inf_{x\in A} \sup_{y\in B} f(x,y).$$

Remark: With $\mathcal{R}(H) = \mathcal{S}(H)$ convex and $\tilde{\mathcal{R}}(L) = \tilde{\mathcal{S}}(L)$ compact, convex, Theorem 1 = Lemma.

$\mathcal{R}(H)$ is w^*-compact convex and $\mathcal{S}(H) = \text{Ex}\mathcal{R}(H)$, the extreme points of $\mathcal{R}(H)$. $\underline{q}(\mathcal{R}(H)) := \underline{q}(\mathcal{R}(H), \text{Ex}\mathcal{R}(H))$, and $q(\mathcal{R}(H)) := q(\mathcal{R}(H), \text{Ex}\mathcal{R}(H))$.

COROLLARY.

With the notation of Theorem 1, let both $\mathcal{R}(H)$ and $\tilde{\mathcal{R}}(L)$ be w^*-compact convex sets of mesures, and let $\mathcal{S}(H)$ and $\tilde{\mathcal{S}}(L)$ be the extreme points of $\mathcal{R}(H)$ and $\tilde{\mathcal{R}}(L)$ respectively. Then

$$\underline{q}\left(\mathcal{R}(H), \tilde{\mathcal{S}}(L)\right) = q\left(\tilde{\mathcal{R}}(L), \mathcal{S}(H)\right),$$

specially if $H = L$, $\mathcal{R}(H) = \tilde{\mathcal{R}}(H)$

$$\underline{q}(\mathcal{R}(H)) = q(\mathcal{R}(H)).$$

Remark: $K \subset\subset X$, $\mathcal{R}(K) = \mathcal{M}_1(K)$ The extremal points of $\mathcal{R}(K)$: Dirac measures concentrated on the points of $K \Rightarrow$ classical potential theory; Farkas-Révész

EXAMPLE.

$f_1, \ldots, f_n$: positive, continuous,

$$\mathcal{M}_{\mathcal{F}}(X) := \{\mu \in \mathcal{M}_1(X) : \int_X f_i d\mu = c_i,\ i = 1, \ldots, n\}.$$

If $\{f_0 \equiv 1, f_1, \ldots, f_n\}$ is a Chebyshev system, then

μ is an extreme point of $\mathcal{M}_{\mathcal{F}}(X)$ $\Leftrightarrow$ card(suppμ) $\leq n+1$
(Douglas, Karr)

Some Chebyshev systems with $f_0 \equiv 1$: $X = [a, b] \subset (0, \infty)$ and $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n$, then $\left\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\right\}$, $\{\cosh \lambda_0 t, \cosh \lambda_1 t, \ldots, \cosh \lambda_n t\}$,

$[a, b] \subset (-\infty, \infty)$, with the same exponents $\left\{e^{\lambda_0 t}, \ldots, e^{\lambda_n t}\right\}$.

Lemma. Let $H \subset X, L \subset Y$ arbitrary and k is a positive l.s.c. symmetric kernel. If for all $\mu \in \mathcal{R}(H)$ and $\varepsilon > 0$ there is a compact set $K(\varepsilon) \in \mathcal{K}(H)$ and a measure $\mu_{K(\varepsilon)}$ such that $\mu_{K(\varepsilon)} \in \mathcal{R}(K(\varepsilon))$ and $\mu_{K(\varepsilon)} \leq (1+\varepsilon)\mu|_{K(\varepsilon)}$, then

$$q(\mathcal{R}(H), \mathcal{S}(L)) = \inf_{K \subset\subset H} q(\mathcal{R}(K), \mathcal{S}(L)).$$

EXAMPLES: (1) $C_p(H)^{-\frac{1}{p}} = \inf_{K \subset\subset H} C_p(K)^{-\frac{1}{p}}$. (2) $\mathcal{R}(H) := \left\{\mu \in \mathcal{M}_1(H) : \int_H f d\mu \geq c\right\}$.

Lemma. Let $H \subset X$ arbitrary and $\mathcal{S}(L)$ is w^*-compact. If for an arbitrary measure $\mu \in \mathcal{R}(H)$ there is a net of measures $\{\mu_K\}_{\mathcal{K}_\mu(H)} \subset \mathcal{R}(H)$ such that $\mathcal{K}_\mu(H) \subset \mathcal{K}(H)$ and $\lim_{K \in \mathcal{K}_\mu(H)} \mu_K = \mu$ in the vague topology, then

$$\underline{q}(\mathcal{R}(H), \mathcal{S}(L)) = \sup_{K \subset\subset H} \underline{q}(\mathcal{R}(K), \mathcal{S}(L)).$$

Lemma. Let $\mathcal{R}(H) \subset \mathcal{M}(H)$ be w^*-compact, convex set of measures. Let us assume that the kernel function is l.s.c. and positive. Then

$$w(\mathcal{R}(H)) \leq q\,(\mathcal{R}(H))\,.$$

Definition. A property fulfils $\mathcal{R}(H)$-nearly everywhere ($\mathcal{R}(H)$-n.e.) on a set H, if denoting by N the set of points of H for which the property does not fulfil, $w(\mathcal{R}(N)) = \infty$.

Lemma. Let $\mu \in \mathcal{M}(X)$, $H \subset X$, $0 \leq t \leq \infty$, k be a symmetric kernel, $\mathcal{R}(H) \subset \mathcal{M}_1(H)$.

If $E(\nu, \mu) \geq t$ for all $\nu \in \mathcal{R}(H)$, $E(\nu) < \infty$, supp$\nu \subset\subset H$, then $U^{\mu}(x) \geq t$ $\mathcal{R}(H)$-n.e. on H.

Definition. Using the notation of the lemma we say that a set of measures $\mathcal{R}(H) \subset \mathcal{M}_1(H)$ is appropriate, if for all $\mu \in \mathcal{R}(H)$ with finite energy such that $E(\nu, \mu) \geq t$ for all $\nu \in \mathcal{R}(H)$, $E(\nu) < \infty$, supp$\nu \subset\subset H$, it fulfils that $U^\mu(x) \geq t$ μ-a.e. on H.

Theorem. Let $H, L \subset X$, $\mathcal{R}(H) \subset \mathcal{M}_1(H)$ be w^*-compact, convex and appropriate set of measures and let $\mathcal{S}(L) \subset \mathcal{M}_1(L)$. Let us assume that the l.s.c. positive symmetric kernel function satisfies the Frostman's maximum principle. Then

$$w(\mathcal{R}(H)) \geq q\left(\mathcal{R}(H), \mathcal{S}(L)\right).$$

COROLLARY

Let $H \subset X$, $\mathcal{R}(H) \subset \mathcal{M}_1(H)$ be appropriate, w^*-compact, convex set of measures and the l.s.c. positive symmetric kernel function k satisfies the Frostman's maximum principle. Then

$$w(\mathcal{R}(H)) = q\,(\mathcal{R}(H))\,.$$

Example.

Let $K \subset\subset X$, (card$K = \infty$) f is a real valued, positive, continuous function on K, The kernel k is positive, symmetric l.s.c. and infinite at the diagonal.

$$\mathcal{R}(K) = \{\mu \in \mathcal{M}_1(K) : \int_X f d\mu = c\}.$$

DISCRETIZATION

Definition. Let $H \subset X$, $L \subset Y$. $\mathcal{R}_n(H) := \{\mu \in \mathcal{R}(H) : \mathrm{card}(\mathrm{supp}\mu) \leq n\}$. The nth Chebyshev constant of $\mathcal{R}(H)$ related to $\mathcal{S}(L)$ is

$$\underline{q}\left(\mathcal{R}_n(H), \mathcal{S}(L)\right),$$

The nth dual Chebyshev constant is

$$q\left(\mathcal{R}_n(H), \mathcal{S}(L)\right).$$

Definition. The Chebyshev constant and the dual Chebyshev constant respectively:

$$\underline{M}\left(\mathcal{R}(H), \mathcal{S}(L)\right) := \lim_{n\to\infty} \underline{q}\left(\mathcal{R}_n(H), \mathcal{S}(L)\right),$$

$$\overline{M}\left(\mathcal{R}(H), \mathcal{S}(L)\right) := \lim_{n\to\infty} q\left(\mathcal{R}_n(H), \mathcal{S}(L)\right).$$

If $\mathcal{R}(H)$ is w^*-compact, convex, let us denote by

$$M\left(\mathcal{R}(H)\right) := M\left(\mathcal{R}(H), \mathsf{Ex}\mathcal{R}(H)\right), \quad \overline{M}\left(\mathcal{R}(H)\right) := \overline{M}\left(\mathcal{R}(H), \mathsf{Ex}\mathcal{R}(H)\right).$$

THEOREM.

Let $H \subset X$, $L \subset Y$ such that $\mathcal{R}(H)$ is w^*-compact, convex and $\mathcal{S}(L)$ is w^*-compact. Then

$$M\left(\mathcal{R}(H), \mathcal{S}(L)\right) = \underline{q}\left(\mathcal{R}(H), \mathcal{S}(L)\right).$$

INFINITE QUADRATIC PROGRAMMING

X, Z: compact metric spaces, $\Phi(x,z) : X \times Z \to \mathbb{R}$, $g : Z \to \mathbb{R}$, $f(x,y) : X \times X \to \mathbb{R}$, $h(x) : X \to \mathbb{R}$: continuous functions, $f(x,y)$: symmetric, positive semi-definite.

$$\inf_{\mu \in \mathcal{R}^+_{\Phi,g}(X)} \frac{1}{2} \int_X \int_X f(x,y) d\mu(x) d\mu(y) + \int_X h(x) d\mu(x),$$

$$\mathcal{R}^+_{\Phi,g}(X) := \{\mu \in \mathcal{M}(X) : \int_X \Phi(x,z) d\mu(x) \geq g(z) \ \forall z \in Z\}.$$

$$\inf_{c_1 \leq c \leq c_2} \inf_{\mu \in \mathcal{R}_{\Phi,g}(X)} c^2 \int_X \int_X k_c(x,y) d\mu(x) d\mu(y) - c^2 K_c,$$

$$\mathcal{R}_{\Phi,g}(X) := \{\mu \in \mathcal{M}_1(X) : \int_X c\Phi(x,z) d\mu(x) \geq g(z) \ \forall z \in Z\},$$

$$k_c(x,y) = \frac{1}{2}\left(f(x,y) + \frac{1}{c}(h(x) + h(y))\right) + K_c.$$

X, Z: compact Hausdorff spaces, $\Phi(x,z) : X \times Z \to \mathbb{R}$, $g : Z \to \mathbb{R}$: continuous functions, $k(x,y)$: positive symmetric lower semicontinuous kernel on $X \times X$.

(CQP$_e$) : $\displaystyle \inf_{\mu \in \mathcal{R}^e_{\Phi,g}(X)} \int_X \int_X k(x,y) d\mu(x) d\mu(y) = w\left(\mathcal{R}^e_{\Phi,g}(X)\right)$, where

$$\mathcal{R}^e_{\Phi,g}(X) := \{\mu \in \mathcal{M}_1(X) : \int_X \Phi(x,z) d\mu(x) = g(z) \ \forall z \in Z\}.$$

The cutting plane algorithm.

Assume: $\mathcal{R}^e_{\Phi,g}(X) \neq \emptyset$.

(1) $n = 1$ choose any $z_1, z_2 \in Z$, $Z_2 = \{z_1, z_2\}$.

(2) $Z_{2n} = \{z_1, z_2, \ldots, z_{2n}\} \Rightarrow \mu_n$: the "minimal measure", $E(\mu_n) = \inf_{\mu \in \mathcal{R}^e_{\Phi,g,Z_{2n}}(X)} E(\mu)$, where $\mathcal{R}^e_{\Phi,g,Z_{2n}}(X) := \{\mu \in \mathcal{M}_1(X) : \int_X \Phi(x,z) d\mu(x) = g(z)\ z \in Z_{2n}\}$

(3) $\Psi_n(z) := \int_X \Phi(x,z) d\mu_n(x) - g(z)$. If $\Psi_n(z) \equiv 0$ then stop. Otherwise determine z_{2n+1}, z_{2n+2}:

$\Psi_n(z_{2n+1}) = \min_{z \in Z} \Psi_n(z)$, $\Psi_n(z_{2n+2}) = \max_{z \in Z} \Psi_n(z)$. $Z_{2n+2} = Z_{2n} \cup \{z_{2n+1}, z_{2n+2}\}$.

Lemma.

Supposing that $\mathcal{R}_{\Phi,g}(X) \neq \emptyset$ $\left(\mathcal{R}^e_{\Phi,g}(X) \neq \emptyset\right)$, the cutting plane algorithm has a limit, that is

$$\lim_{n\to\infty} w\left(\mathcal{R}^e_{\Phi,g,Z_{2n}}(X)\right) = w\left(\mathcal{R}^e_{\Phi,g}(X)\right).$$

Lemma. Let $f_1, \ldots, f_n$ be positive, bounded, continuous, totally non-constant functions on X, k is a symmetric l.s.c. kernel, let c_i be positive numbers, $i = 1, \ldots, n$. Assume that $\mathcal{M}_{\mathcal{F}}(X) := \{\mu \in \mathcal{M}_1(X) : \int_X f_i d\mu = c_i \, i = 1, \ldots, n\} \neq \emptyset$. If k is infinite at the diagonal or $1, f_1, \ldots, f_n$ is a strong Chebyshev (or Descartes) system, then $\mathcal{M}_{\mathcal{F}}(X)$ is appropriate.

Let $\mathcal{R}(X) := \mathcal{R}^e_{\Phi,g}(X)$, and $\mathcal{R}^{(n)}(X) := \mathcal{R}^e_{\Phi,g,Z_n}(X)$,

$\mathcal{R}^{(n)}_m(X) := \{\mu \in \mathcal{R}^{(n)}(X) : \text{Card}(\text{supp}\mu) \leq m\}$.

Theorem.

Besides the assumptions of (CQP) and Lemma 10, let us suppose that the kernel $k(x, y)$ fulfils the Frostman's maximum principle. Then

$$(CQP) = \inf_{\mu \in \mathcal{R}(X)} \int_X \int_X k(x,y) d\mu(x) d\mu(y) = w(\mathcal{R}(X)) = \lim_{n\to\infty} w(\mathcal{R}^{(n)}(X))$$

$$= \lim_{n\to\infty} q(\mathcal{R}^{(n)}(X)) = \lim_{n\to\infty} \inf_{\mu \in \mathcal{R}^{(n)}(X)} \sup_{\nu \in \mathsf{Ex}\mathcal{R}^{(n)}(X)} \int_X \int_X k(x,y) d\mu(x) d\nu(y)$$

$$= \lim_{n\to\infty} \underline{q}(\mathcal{R}^{(n)}(X)) = \lim_{n\to\infty} \sup_{\mu \in \mathcal{R}^{(n)}(X)} \inf_{\nu \in \mathsf{Ex}\mathcal{R}^{(n)}(X)} \int_X \int_X k(x,y) d\mu(x) d\nu(y)$$

$$= \lim_{n\to\infty} M(\mathcal{R}^{(n)}(X)) = \lim_{n\to\infty} \lim_{m\to\infty} \sup_{\mu \in \mathcal{R}_m^{(n)}(X)} \inf_{\nu \in \mathsf{Ex}\mathcal{R}^{(n)}(X)} \int_X \int_X k(x,y) d\mu(x) d\nu(y)$$

$$= \lim_{n\to\infty} \lim_{m\to\infty} \underline{q}(\mathcal{R}_m^{(n)}(X), \mathsf{Ex}\mathcal{R}^{(n)}(X)).$$

Corollary.

If X is a compact Hausdorff space, k is l.s.c. and fulfils the Frostman's maximum principle, $\mathcal{F} \subset L^1_\mu \ \forall \mu \in \mathcal{M}_1(X)$ and is a Chebyshev system, $\mathcal{F}_n = \{f_0, f_1, \ldots, f_n\} \subset \mathcal{F}$, then

$$CQP = w(\mathcal{R}_{\Phi,g}(X))$$

$$= \lim_{n\to\infty} \inf_{\substack{\mu, \int_X f_i d\mu = c_i, \\ i=0,1,\ldots,n}} \sup_{\substack{y_1,\ldots,y_l \in X,\ l \le n+1,\ \alpha_j > 0 \\ \sum_{j=1}^l \alpha_j f_i(y_j) = c_i, i=0,1,\ldots,n}} \sum_{j=1}^{l} \alpha_j \int_X k(x, y_j) d\mu(x)$$

$$= \lim_{n\to\infty} \lim_{m\to\infty} \sup_{\substack{x_1,\ldots,x_m \in X,\ \beta_k > 0 \\ \sum_{k=1}^m \beta_k f_i(x_k) = c_i, i=0,1,\ldots,n}} \inf_{\substack{y_1,\ldots,y_l \in X,\ l \le n+1,\ \alpha_j > 0 \\ \sum_{j=1}^l \alpha_j f_i(y_j) = c_i, i=0,1,\ldots,n}} \sum_{j=1}^{l} \sum_{k=1}^{m} \alpha_j \beta_k k(x_k, y_j).$$

Example. Let $H = K = [0,1]$ (as subsets of a suitable compact subset of $\mathbb{C}$, say) For instance let $k(x,y) = \log \frac{1}{|x-y|}$. Let $\{c_i\}_{i=1}^N$ such that $(-1)^r \Delta^r c_k \geq 0$ for each $r, k \leq N$, $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_M$, $(d_1, \ldots, d_M) \in \mathrm{co}\Gamma u$ as above. Then

$$\sup_{\substack{\mu \\ \int_0^1 x^i d\mu = c_i,\ i=0,\ldots,N}} \inf_{\substack{\sum_{k=0}^{l} \alpha_k \delta_{y_k},\ y_k \in [0,1] \alpha_k > 0,\ l \leq M \\ \sum_{k=0}^{l} \alpha_k e^{-\lambda_j y_k} = d_j,\ j=0,\ldots,M}} \sum_{k=0}^{l} \alpha_k \int_0^1 k(x, y_k) d\mu(x)$$

$$= \inf_{\substack{\nu \\ \int_0^1 e^{-\lambda_j y} d\nu(y) = d_j,\ j=0\ldots,M}} \sup_{\substack{\sum_{r=0}^{s} \beta_r \delta_{x_r}\ x_r \in [0,1] \beta_r > 0,\ s \leq N \\ \sum_{r=0}^{s} \beta_r x_r^i = c_i,\ i=0,\ldots,N}} \sum_{r=0}^{s} \beta_r \int_0^1 k(x_r, y) d\nu(y).$$