Lineáris komplementaritási feladatok: elmélet, algoritmusok, alkalmazások

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Outline

- Linear complementarity problem (LCP):
  $-M\,\mathbf{u} + \mathbf{v} = \mathbf{q}, \quad \mathbf{u} \geq \mathbf{0},\ \mathbf{v} \geq \mathbf{0}, \quad \mathbf{u}\,\mathbf{v} = \mathbf{0}$
- Motivations:
  linear programming -, linearly constrained, convex quadratic programming -, and discrete knapsack problems
- Matrix classes
- LCP duality theory and EP theorems
- Variants of the criss-cross algorithm
  - general LCP - cycling
- Interior point algorithms for LCP
  - central path, Newton-system, scaling, proximity measures
  - variants of IPAs
  - example of an IPA
  - polynomial size certificates of lack of sufficiency

Primal-dual linear programming problems

$$\left.\begin{array}{rcl} \min & \mathbf{c}^T\mathbf{x} & \\ A\mathbf{x} & \geq & \mathbf{b} \\ \mathbf{x} & \geq & \mathbf{0} \end{array}\right\} \quad (P) \qquad\qquad \left.\begin{array}{rcl} \max & \mathbf{b}^T\mathbf{y} & \\ A^T\mathbf{y} & \leq & \mathbf{c} \\ \mathbf{y} & \geq & \mathbf{0} \end{array}\right\} \quad (D)$$

Theorem (Weak duality theorem.)

For any $\mathbf{x}$ *primal- and* $\mathbf{y}$ *dual feasible solution* $\mathbf{c}^T\mathbf{x} \geq \mathbf{b}^T\mathbf{y}$ *and equality holds if and only if* $\mathbf{x}^T\mathbf{s}+\mathbf{y}^T\mathbf{z}=\mathbf{0}$, *where* $\mathbf{z}=A\mathbf{x}-\mathbf{b}$ *and* $\mathbf{s}=\mathbf{c}-A^T\mathbf{y}$.

Optimality conditions.

$$\begin{array}{rrrrcrl} & -A\mathbf{x}+ & \mathbf{z} & & = & -\mathbf{b}, & \mathbf{x}\geq \mathbf{0},\ \ \mathbf{z}\geq \mathbf{0}, \\ A^T\mathbf{y}+ & & & \mathbf{s} & = & \mathbf{c}, & \mathbf{y}\geq \mathbf{0},\ \ \mathbf{s}\geq \mathbf{0}, \\ & & \mathbf{x}\,\mathbf{s}+ & \mathbf{y}\,\mathbf{z} & = & \mathbf{0}. & \end{array}$$

$$M=\left[\begin{array}{cc} 0 & A \\ -A^T & 0 \end{array}\right],\ \mathbf{u}=\begin{pmatrix}\mathbf{y}\\ \mathbf{x}\end{pmatrix},\ \mathbf{v}=\begin{pmatrix}\mathbf{z}\\ \mathbf{s}\end{pmatrix},\ \mathbf{q}=\begin{pmatrix}-\mathbf{b}\\ \mathbf{c}\end{pmatrix}$$

Linearly constrained, quadratic optimization problem ($LCQOP$)

$$\min \frac{1}{2}\mathbf{x}^T Q\mathbf{x}+\mathbf{c}^T\mathbf{x}$$
$$A\mathbf{x}\leq \mathbf{b}, \quad \mathbf{x}\geq \mathbf{0}$$

where $Q\in\mathbb{R}^{n\times n}$ and $A\in\mathbb{R}^{m\times n}$ are matrices, while $\mathbf{c}\in\mathbb{R}^n$ and $\mathbf{b}\in\mathbb{R}^m$ are vectors. Without loss of generality we may assume that $rank(A)=m$.

Decision variables: $\mathbf{x}\in\mathbb{R}^n$

Objective function of the ($LCQOP$): $f(\mathbf{x})=\frac{1}{2}\mathbf{x}^T Q\mathbf{x}+\mathbf{c}^T\mathbf{x}$

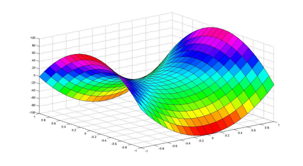
Feasible solution set: $\mathcal{P}=\{\mathbf{x}\in\mathbb{R}^n : A\mathbf{x}\leq\mathbf{b},\ \mathbf{x}\geq\mathbf{0}\}\subset\mathbb{R}^n$

Optimal solution set:

$$\mathcal{P}^*=\{\mathbf{x}^*\in\mathcal{P} : f(\mathbf{x}^*)\leq f(\mathbf{x}) \quad \text{holds for any} \ \ \mathbf{x}\in\mathcal{P}\}$$

$(LCQOP)$: basic properties, difficulties

Function $f : \mathcal{P} \to \mathbb{R}$ given as $f(\mathbf{x}) = \frac{1}{2}\,\mathbf{x}^T Q\mathbf{x} + \mathbf{c}^T\mathbf{x}$ is a continuous, quadratic function.
If $\mathcal{P}$ is bounded then $(LCQOP)$ has minimum.



Example.

$f(x_1, x_2) = 100\,x_1^2 - 100\,x_2^2 + 2\,x_1\,x_2$

$-1 \leq x_1, x_2 \leq 1$

Definition. A $Q \in \mathbb{R}^{m\times m}$ matrix, is called *positive semidefinite matrix*, if for any $\mathbf{x} \in \mathbb{R}^m$ vector $\mathbf{x}^T Q\mathbf{x} \geq 0$ holds. •

The Lagrange-function of $(LCQOP)$ problem $L : \mathbb{R}^n \times \mathbb{R}^{m+n}_{\oplus} \to \mathbb{R}$ has been defined as follows:

$$L(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \frac{1}{2}\,\mathbf{x}^T Q\mathbf{x} + \mathbf{c}^T\mathbf{x} + \mathbf{y}^T(A\mathbf{x} - \mathbf{b}) - \mathbf{z}^T\mathbf{x}.$$

Convex QP: Karush-Kuhn-Tucker theorem

Theorem. Let the $(LCQOP)$ problem be given. A vector $\mathbf{x}^* \in \mathcal{P}^*$ if and only if, there exist $\mathbf{y}^* \in \mathbb{R}^m_\oplus, \mathbf{z}^* \in \mathbb{R}^n_\oplus$ such that $(\mathbf{y}^*, \mathbf{z}^*) \neq \mathbf{0}$ and $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*)$ satisfies the *Karush-Kuhn-Tucker system*

$$
\begin{aligned}
Q\mathbf{x} + c + A^T\mathbf{y} - \mathbf{z} &= \mathbf{0}, & A\mathbf{x} - \mathbf{b} &\leq \mathbf{0} \\
\mathbf{y}^T(A\mathbf{x} - \mathbf{b}) &= 0 & \mathbf{z}^T\mathbf{x} &= 0 \\
\mathbf{x}, \mathbf{y}, \mathbf{z} &\geq \mathbf{0}.
\end{aligned}
$$

Introducing slack variable $\mathbf{s} \in \mathbb{R}^m_\oplus$, the previous KKT-system (*optimality criteria for convex QP*) can be rewritten as follows:

$$
\begin{aligned}
-Q\mathbf{x} - A^T\mathbf{y} + \mathbf{z} &= \mathbf{c}, & A\mathbf{x} + \mathbf{s} &= \mathbf{b} \\
\mathbf{y}^T\mathbf{s} &= 0 & \mathbf{z}^T\mathbf{x} &= 0 \\
\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{s} &\geq \mathbf{0}.
\end{aligned}
$$

Convex QP vs. linear complementarity problem

The linearly constrained, convex quadratic programming problem is equivalent to the following linear complementarity problem (LCP_{QP}):

$$-M\mathbf{u} + \mathbf{v} = \mathbf{q}, \quad \mathbf{u}^T\mathbf{v} = 0, \quad \mathbf{u}, \mathbf{v} \geq 0, \quad \text{where}$$

$$M = \begin{pmatrix} Q & A^T \\ -A & 0 \end{pmatrix}, \;\; \mathbf{q} = \begin{pmatrix} \mathbf{c} \\ \mathbf{b} \end{pmatrix}, \;\; \text{and} \;\; \mathbf{v} = \begin{pmatrix} \mathbf{z} \\ \mathbf{s} \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}.$$

Solution set: $\mathcal{F} = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^{2N} \,:\, -M\,\mathbf{u} + \mathbf{v} = \mathbf{q}\}$.

Feasible solution set: $\mathcal{F}_{\oplus} = \{(\mathbf{u}, \mathbf{v}) \in \mathcal{F} \,:\, \mathbf{u} \geq \mathbf{0},\ \mathbf{v} \geq \mathbf{0}\}$.

Complementarity solution set: $\mathcal{F}_c = \{(\mathbf{u}, \mathbf{v}) \in \mathcal{F} \,:\, \mathbf{u}\,\mathbf{v} = \mathbf{0}\}$.

Complementarity feasible solution set: $\mathcal{F}^* = \mathcal{F}_{\oplus} \cap \mathcal{F}_c$.

Linearly constrained, convex quadratic programming

$$\left.\begin{array}{rcccccc} \min & \mathbf{c}^T\mathbf{x} & + & \frac{1}{2}\,\mathbf{x}^T C^T C\,\mathbf{x} & + & \frac{1}{2}\,\mathbf{z}^T\mathbf{z} \\ & A\mathbf{x} & + & B\mathbf{z} & \geq & \mathbf{b} \\ & & & \mathbf{x} & \geq & \mathbf{0} \end{array}\right\} \quad (QP),$$

$$\left.\begin{array}{rcccccc} \max & \mathbf{y}^T\mathbf{b} & - & \frac{1}{2}\,\mathbf{y}^T B B^T\mathbf{y} & - & \frac{1}{2}\,\mathbf{w}^T\mathbf{w} \\ & \mathbf{y}^T A & - & \mathbf{w}^T C & \leq & \mathbf{c} \\ & & & \mathbf{y} & \geq & \mathbf{0} \end{array}\right\} \quad (QD),$$

$A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times k}$, $C \in \mathbb{R}^{l\times n}$, $\mathbf{c}, \mathbf{x} \in \mathbb{R}^n$, $\mathbf{b}, \mathbf{y} \in \mathbb{R}^m$, $\mathbf{z} \in \mathbb{R}^k$, $\mathbf{w} \in \mathbb{R}^l$.

Theorem (Weak duality theorem.)

For any $(\mathbf{x}, \mathbf{z})$ primal- and $(\mathbf{y}, \mathbf{w})$ dual feasible solution

$$\mathbf{c}^T\mathbf{x} + \frac{1}{2}\,\mathbf{x}^T C^T C\,\mathbf{x} + \frac{1}{2}\,\mathbf{z}^T\mathbf{z} \;\geq\; \mathbf{y}^T\mathbf{b} - \frac{1}{2}\,\mathbf{y}^T B B^T\mathbf{y} - \frac{1}{2}\,\mathbf{w}^T\mathbf{w},$$

and equality holds if and only if $\mathbf{w} = C\mathbf{x}$ and $\mathbf{z} = B^T\mathbf{y}$.

Linear complementarity problem with bisymmetric matrix

Problem (LCP_{QP})

Klafszky, Terlaky (1992)

$$\begin{array}{rcrcrcrcl}
-P\mathbf{y} & - & A\mathbf{x} & + & \bar{\mathbf{y}} & & & = & -\mathbf{b} \\
A^T\mathbf{y} & - & Q\mathbf{x} & & & + & \bar{\mathbf{x}} & = & \mathbf{c} \\
& & \mathbf{x}, & \mathbf{y}, & \bar{\mathbf{x}}, & \bar{\mathbf{y}} & & \geq & \mathbf{0} \\
& & \mathbf{x}\,\bar{\mathbf{x}} & = & 0, & & \mathbf{y}\,\bar{\mathbf{y}} & = & 0
\end{array}$$

$P = BB^T$ and $Q = C^TC$ are positive semidefinite matrices.

$$M = \begin{bmatrix} P & A \\ -A^T & Q \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} + \begin{bmatrix} 0 & A \\ -A^T & 0 \end{bmatrix},$$

$$\mathbf{u} = \begin{pmatrix} \mathbf{y} \\ \mathbf{x} \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} \bar{\mathbf{y}} \\ \bar{\mathbf{x}} \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} -\mathbf{b} \\ \mathbf{c} \end{pmatrix}$$

Theorem

The (QP) and (QD) problems have optimal solution if and only if the (LCP_{QP}) problem has feasible and complementary solution.

Bimatrix games

- Players: P_1 and P_2. Finite set of strategies: $\mathcal{I} = \{1, 2, \dots, n\}$ and $\mathcal{J} = \{1, 2, \dots, m\}$.
- Payoff matrices: $A, B \in \mathbb{R}^{n \times m}$ (assumption: all entries are positive).
- Payoff rule: if they play the $i \in \mathcal{I}$ and $j \in \mathcal{J}$ strategies respectively, then the first player pays a_{ij} amount of money and the second player pays b_{ij}.
- Mixed strategies:

$$\mathcal{S}_n = \{\mathbf{x} \in \mathbb{R}^n_\oplus : \mathbf{e}^T\mathbf{x} = 1\} \quad \text{and} \quad \mathcal{S}_m = \{\mathbf{y} \in \mathbb{R}^m_\oplus : \mathbf{e}^T\mathbf{y} = 1\},$$

  where $\mathbf{e}$ is a vector of all 1's with proper size.
- Expected costs of the game:

$$E_1(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T A\mathbf{y} \quad \text{and} \quad E_2(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T B\mathbf{y}.$$

- Each player would like to minimize his payoff.

Nash equilibrium

A pair of mixed strategies $(\mathbf{x}^*, \mathbf{y}^*)$ is a *Nash equilibrium* if

$$
\begin{array}{lll}
E_1(\mathbf{x}^*, \mathbf{y}^*) \leq E_1(\mathbf{x}, \mathbf{y}^*) & (\mathbf{x}^*)^T\ A\,\mathbf{y}^* \leq \mathbf{x}^T\ A\,\mathbf{y}^* & \text{for all } \mathbf{x} \in \mathcal{S}_n, \\
E_2(\mathbf{x}^*, \mathbf{y}^*) \leq E_2(\mathbf{x}^*, \mathbf{y}) & (\mathbf{x}^*)^T\ B\,\mathbf{y}^* \leq (\mathbf{x}^*)^T\ B\,\mathbf{y} & \text{for all } \mathbf{y} \in \mathcal{S}_m.
\end{array}
$$

Computing a Nash equilibrium point can be expressed as solving an (LCP) of the following form:

$$
\left.\begin{array}{lll}
\mathbf{u} = -\mathbf{e} + A\,\mathbf{y} \geq \mathbf{0}, & \mathbf{x} \geq \mathbf{0}, & \mathbf{x}^T\mathbf{u} = 0 \\
\mathbf{v} = -\mathbf{e} + B^T\mathbf{x} \geq \mathbf{0}, & \mathbf{y} \geq \mathbf{0}, & \mathbf{y}^T\mathbf{v} = 0
\end{array}\right\}, \qquad (1)
$$

namely

$$
M = \begin{bmatrix} O & A \\ B^T & O \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}, \qquad \mathbf{q} = -\mathbf{e} \in \mathbb{R}^{n+m}.
$$

The matrix M has only non-negative entries.

Bimatrix games – LCP

Theorem

Let us assume that a bimatrix game is given with the players finite sets of strategies, $\mathcal{I}$ and $\mathcal{J}$ and payoff matrices, A and B. Furthermore, let us assume that the corresponding (LCP) has been formulated as (1).

- *Let $(\mathbf{x}^*, \mathbf{y}^*)$ be a Nash equilibrium of the bimatrix game, then*
  $$\mathbf{x}' = \frac{\mathbf{x}^*}{(\mathbf{x}^*)^T B\,\mathbf{y}^*} \quad \text{and} \quad \mathbf{y}' = \frac{\mathbf{y}^*}{(\mathbf{x}^*)^T A\,\mathbf{y}^*} \quad \text{is a solution of}$$
  problem (1).
- *Let $(\mathbf{x}', \mathbf{y}')$ be a solution of the (LCP) defined by (1), then*
  $$\mathbf{x}^* = \frac{\mathbf{x}'}{\mathbf{e}^T\mathbf{x}'} \quad \text{and} \quad \mathbf{y}^* = \frac{\mathbf{y}'}{\mathbf{e}^T\mathbf{y}'} \quad \text{is a Nash equilibrium.}$$

Léon Walras (1874)

Exchange market equilibrium problem:

- there are m traders (players) and n goods on the market,
- each good j has a price $p_j \geq 0$,
- each trader i has an initial endowment of commodities $\mathbf{w}^i = (w_{i1}, \ldots, w_{in}) \in \mathbb{R}^n_{\oplus}$,
- traders sell their product on the market and use their income to buy a bundle of goods $\mathbf{x}^i = (x_{i1}, \ldots, x_{in}) \in \mathbb{R}^n_{\oplus}$,
- each trader i has
  - a utility function u_i, which describes his preferences for the different bundle of commodities,
  - a budget constraint $\mathbf{p}^T\mathbf{x}^i \leq \mathbf{p}^T\mathbf{w}^i$,
- each trader i maximizes his individual utility function subject to his budget constraint.

Exchange market equilibrium problem

The vector of prices $\mathbf{p}$ is an *equilibrium* for the exchange economy, if there is a bundle of goods $\mathbf{x}^i(\mathbf{p})$ (so a maximizer of the utility function u_i subject to the budget constraint) for all traders i, such that

$$\sum_{i=1}^{m} x_{ij}(\mathbf{p}) \leq \sum_{i=1}^{m} w_{ij} \quad \text{for all goods } j.$$

Walras asked the following question: Are there such prices of goods, where the demand $\sum_i x_{ij}(\mathbf{p})$ does not exceed the supply $\sum_i w_{ij}$ for all good j? Do equilibrium prices exist for the exchange economy?

Namely, whether the prices for goods could be set in such a way that each trader can maximize his utility function individually.

Arrow and Debreu (1954) proved that under mild conditions, for concave utility functions, the exchange markets equilibrium exists.

Arrow–Debreu model & special LCP

Leontief utility functions:

$$u_i(\mathbf{x}^i) = \min_j \left\{ \frac{x_{ij}}{a_{ij}} : \ a_{ij} > 0 \right\},$$

where $A = (a_{ij}) \in \mathbb{R}^{n \times n}_{\oplus}$ is the Leontief coefficient matrix. We assume that every trader likes at least one commodity, so the matrix A has no all-zero row.

Solution of the Arrow–Debreu competitive market equilibrium problem with Leontief's utility function is equivalent to the following *linear complementarity problem* $(LCP_{AD-pc-Luf})$:

$$A^T\,\mathbf{u} + \mathbf{v} = \mathbf{e}, \quad \mathbf{u} \geq \mathbf{0},\ \mathbf{v} \geq \mathbf{0}, \quad \mathbf{u}\,\mathbf{v} = \mathbf{0} \ \text{ and } \ \mathbf{u} \neq \mathbf{0}$$

where the matrix A has non-negative entries [Ye (2007)]. Using our original (LCP) notation, $M = -A^T$, thus each entry of M is non-positive.

Summary of LCP examples

Linear complementarity problems (LCP):

$$-M\,\mathbf{u} + \mathbf{v} = \mathbf{q}, \quad \mathbf{u} \geq \mathbf{0},\, \mathbf{v} \geq \mathbf{0}, \quad \mathbf{u}\,\mathbf{v} = \mathbf{0}$$

Problem	properties of matrix M
Primal-dual LP problem	skew symmetric
Primal-dual, linearly constrained, convex QP problem	bisymmetric (diagonal blocks: PSD, and off-diagonal: skew symmetric)
Discrete Knapsack Problem	lower triangular: -1's in the diagonal
Markov chain problem	non-negative diagonal and non-positive off-diagonal
Bimatrix games	non-negative entries ($\mathbf{q} = -\mathbf{e}$)
Arrow–Debreu problem	non-positive entries ($\mathbf{q} = \mathbf{e}, \mathbf{u} \neq \mathbf{0}$)

LCP – matrix classes

Let M, $X = diag(\mathbf{x}) \in \mathbb{R}^{n \times n}$ matrices, for all $\mathbf{x} \in \mathbb{R}^n$:

skew symmetric: $\quad \mathrm{x}^T M\,\mathbf{x} = 0$

PSD (PD): $\quad \mathbf{x}^T M\,\mathbf{x} \geq 0\ (\mathbf{x}^T M\,\mathbf{x} > 0,$ where $\mathbf{x} \neq \mathbf{0})$

P (P_0): $\quad$ all principal minors are positive (nonnegative)

CS: $\quad X(M\,\mathbf{x}) \leq \mathbf{0} \quad \Rightarrow \quad X(M\,\mathbf{x}) = \mathbf{0}.$

Cottle, Pang, Venkateswaran (1989)

RS: $\quad M^T$ is column sufficient.

S: $\quad M$ is column and row sufficient.

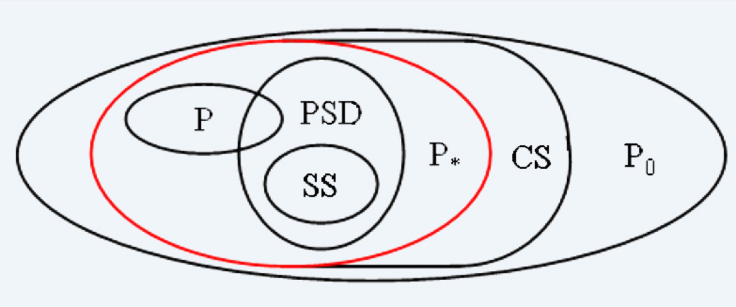
Kojima, Megiddo, Noma, Yoshise (1991)

$$(1+4\,\kappa) \sum_{i \in I_+(x)} x_i (M\,\mathbf{x})_i + \sum_{i \in I_-(x)} x_i (M\mathbf{x})_i \geq 0 \ \text{ and } \ \mathcal{P}_* = \bigcup_{\kappa \geq 0} \mathcal{P}_*(\kappa)$$

where $I_+(x) = \{i\ :\ x_i(M\,\mathbf{x})_i > 0\}$ and $I_-(x) = \{i\ :\ x_i(M\,\mathbf{x})_i < 0\}$.

LCP – matrix classes II.

- Kojima, Megiddo, Noma, Yoshise (1991): $\mathcal{P}_* \subseteq \mathcal{CS}$
- Guu and Cottle (1995): $\mathcal{P}_* \subseteq \mathcal{RS}$
- Väliaho (1996): $\mathcal{S} \subseteq \mathcal{P}_*$



LCP – matrix classes: example

Example. Let $a, b > 0$ real numbers, satisfying $a\,b = 2$ and

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & a \\ 0 & -b & 0 \end{pmatrix} \qquad \text{and} \qquad X\,M\,\mathbf{x} = \begin{pmatrix} x_1^2 \\ x_2\,(x_1 + 2\,x_2 + a\,x_3) \\ -b\,x_2\,x_3 \end{pmatrix}$$

It is easy to show that M is a P_0-matrix, with eigenvalues $1,\ 1+i,\ 1-i$.

There is no $\mathbf{x} \in \mathbb{R}^3$ such that $X\,M\,\mathbf{x} \leq \mathbf{0}$ holds, $\rightsquigarrow$ M is **column sufficient** matrix. (Row sufficiency can be checked similarly.)

$$\kappa(M) = ?$$

Theorem — Tseng (2000)

Let an integer square matrix M be given, the decision problem for matrix class P, P_0 and for sufficient matrices is co-$\mathbb{NP}$-complete.

LCP – primal and dual problems

The (primal) linear complementarity problem $(P-LCP)$ is given above.

$$\mathcal{F}_{\oplus} = \left\{(\mathbf{u},\mathbf{v}) \in \mathbb{R}^{2n} : -M\,\mathbf{u}+\mathbf{v}=\mathbf{q},\ \mathbf{u},\mathbf{v}\geq \mathbf{0}\right\}$$

$$\mathcal{F}_{c} = \left\{(\mathbf{u},\mathbf{v}) \in \mathbb{R}^{2n} : -M\,\mathbf{u}+\mathbf{v}=\mathbf{q},\ \ \mathbf{u}\,\mathbf{v}=\mathbf{0}\right\}$$

$$\mathcal{F}_{+} = \mathcal{F}_{\oplus}\cap \mathbb{R}^{2n}_{+} \qquad \text{and} \qquad \mathcal{F}^{*} = \mathcal{F}_{\oplus}\cap \mathcal{F}_{c}$$

The (dual) linear complementarity problem $(D-LCP)$ is

$$\mathbf{x}+M^{T}\mathbf{y}=\mathbf{0},\ \ \mathbf{q}^{T}\mathbf{y}=-1,\ \ \mathbf{x},\,\mathbf{y}\geq \mathbf{0},\ \ \mathbf{x}\,\mathbf{y}=\mathbf{0}.$$

$$\mathcal{D}_{\oplus} = \{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2n} : \mathbf{x}+M^{T}\mathbf{y}=\mathbf{0},\ \ \mathbf{q}^{T}\mathbf{y}=-1,\ \ \mathbf{x},\,\mathbf{y}\geq \mathbf{0}\}$$

$$\mathcal{D}_{C} = \{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2n} : \mathbf{x}+M^{T}\mathbf{y}=\mathbf{0},\ \ \mathbf{q}^{T}\mathbf{y}=-1,\ \ \mathbf{x}\,\mathbf{y}=\mathbf{0}\}$$

$$\mathcal{D}_{+} = \mathcal{D}_{\oplus}\cap \mathbb{R}^{2n}_{+} \qquad \text{and} \qquad \mathcal{D}^{*} = \mathcal{D}_{\oplus}\cap \mathcal{D}_{C}$$

LCP duality theory (or alternative theorem for LCP)

$$\mathbf{x}+M^T\mathbf{y}=\mathbf{0},\ \ \mathbf{q}^T\mathbf{y}=-1,\ \ \mathbf{x},\,\mathbf{y}\geq\mathbf{0},\ \ \mathbf{x}\,\mathbf{y}=\mathbf{0}.$$

Both can't have a solution. Let us assume contrary that both can have a solution, then

$$\mathbf{y}^T(-M\mathbf{u}+\mathbf{v})=-\mathbf{y}^TM\mathbf{u}+\mathbf{y}^T\mathbf{v}=\mathbf{y}^T\mathbf{q}=-1$$

$$\mathbf{u}^T(\mathbf{x}+M^T\mathbf{y})=\mathbf{u}^T\mathbf{x}+\mathbf{u}^TM^T\mathbf{y}=\mathbf{u}^T\mathbf{0}=0$$

$\rightsquigarrow\ \ 0\leq\mathbf{u}^T\mathbf{x}+\mathbf{y}^T\mathbf{v}=-1$ we have got a contradiction.

Theorem Fukuda, Terlaky (1992)

For a sufficient matrix $M\in\mathbb{R}^{n\times n}$, and a vector $\mathbf{q}\in\mathbb{R}^n$, exactly one of the following statements hold:

$$(1)\quad \mathcal{F}^*\neq\emptyset, \qquad\qquad (2)\quad \mathcal{D}^*\neq\emptyset.$$

Constructive proof by minimal index criss-cross algorithm.

Linear complementarity problems

Proposition

Let $\mathcal{F}_+ \neq \emptyset$ and $M \in \mathcal{P}_*(\kappa)$ then $\mathcal{F}^* \neq \emptyset$, convex, closed and bounded set.

$$\mathbf{x} + M^T\mathbf{y} = \mathbf{0}, \;\; \mathbf{q}^T\mathbf{y} = -1, \;\; \mathbf{x}, \mathbf{y} \geq \mathbf{0}, \;\; \mathbf{x}\,\mathbf{y} = \mathbf{0}.$$

Lemma

Illés, M. Nagy, Terlaky (2007)

Let M be row sufficient and $\mathbf{q} \in \mathbb{R}^n$. If $(\mathbf{x}, \mathbf{y}) \in \mathcal{D}_\oplus$, then $(\mathbf{x}, \mathbf{y}) \in \mathcal{D}^*$, thus $\mathcal{D}^* = \mathcal{D}_\oplus \neq \emptyset$, closed, convex polyhedron. Furthermore, the dual LCP can be solved in polynomial time.

EP theorem and LCP duality in EP-form

Theorem (General form of an EP theorem) Cameron, Edmonds (1990)

$$[\forall \mathbf{x} : F_1(\mathbf{x}) \text{ or } F_2(\mathbf{x}) \text{ or } \dots \text{ or } F_k(\mathbf{x})]$$

where $F_i(\mathbf{x})$ is a statement of the form

$$F_i(\mathbf{x}) = \left[\exists \mathbf{y}_i \text{ for which } \|\mathbf{y}_i\| \leq \|\mathbf{x}\|^{n_i} \text{ and } f_i(\mathbf{x}, \mathbf{y}_i)\right].$$

Theorem Fukuda, Namiki, Tamura (1998)

For any matrix $M \in \mathbb{Q}^{n\times n}$ and vector $\mathbf{q} \in \mathbb{Q}^n$, at least one of the following statements holds:

(1) $\exists\,(\mathbf{u},\mathbf{v}) \in \mathcal{F}^*$ *whose encoding size is polynomially bounded,*

(2) $\exists\,(\mathbf{x},\mathbf{y}) \in \mathcal{D}^*$ *whose encoding size is polynomially bounded,*

(3) *the matrix M is not sufficient, and there is a certificate whose encoding size is polynomially bounded.*

Constructive proof by minimal index criss-cross algorithm.

LCP duality in EP-form II.

Theorem (EP 1)

Illés, M. Nagy, Terlaky (2008)

For any matrix $M \in \mathbb{Q}^{n \times n}$ and vector $\mathbf{q} \in \mathbb{Q}^n$, it can be shown in polynomial time that at least one of the following statements holds:

(1) $\exists\,(\mathbf{x}, \mathbf{y}) \in \mathcal{D}^*$, *whose encoding size is polynomially bounded,*

(2) $\exists\,(\mathbf{u}, \mathbf{v}) \in \mathcal{F}_\oplus$, *whose encoding size is polynomially bounded,*

(3) *the matrix M is not row sufficient and there is a certificate whose encoding size is polynomially bounded.*

Theorem (EP 2)

Illés, M. Nagy, Terlaky (2008)

For any matrix $M \in \mathbb{Q}^{n \times n}$ and vector $\mathbf{q} \in \mathbb{Q}^n$, it can be shown in polynomial time that at least one of the following statements holds:

(1) $\exists\,(\mathbf{u}, \mathbf{v}) \in \mathcal{F}^*$ *whose encoding size is polynomially bounded,*

(2) $\exists\,(\mathbf{x}, \mathbf{y}) \in \mathcal{D}^*$ *whose encoding size is polynomially bounded,*

(3) $M \notin \mathcal{P}_*(\tilde{\kappa})$, *for a given* $\tilde{\kappa} > 0$.

Criss-Cross algorithm - example

$$-M\,\mathbf{u} + \mathbf{v} = \mathbf{q},\quad \mathbf{u} \geq \mathbf{0},\ \mathbf{v} \geq \mathbf{0},\quad \mathbf{u}\,\mathbf{v} = \mathbf{0}$$

Initial tableau, pivot in infeasible row

	u_1	u_2	u_3	v_1	v_2	v_3	
v_1	-1	1	0	1	0	0	0
v_2	0	0	1	0	1	0	0
v_3	0	-1	0	0	0	1	-1

$\rightarrow$

Exchange pivot pair

	u_1	u_2	u_3	v_1	v_2	v_3	
v_1	-1	0	0	1	0	1	-1
v_2	0	0	1	0	1	0	0
u_2	0	1	0	0	0	-1	1

$\swarrow$

Diagonal pivot

	u_1	u_2	u_3	v_1	v_2	v_3	
v_1	-1	0	0	1	0	1	-1
u_3	0	0	1	0	1	0	0
u_2	0	1	0	0	0	-1	1

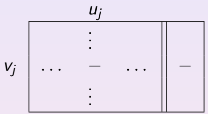
$\rightarrow$

Feasible tableau

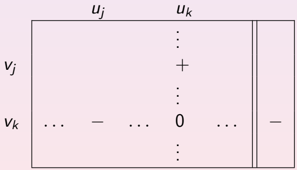
	u_1	u_2	u_3	v_1	v_2	v_3	
u_1	1	0	0	-1	0	-1	1
u_3	0	0	1	0	1	0	0
u_2	0	1	0	0	0	-1	1

Eigenvalues of M are $1, i, -i$. M is a P_0 matrix, however, M is NOT a sufficient matrix, certificate is $\mathbf{x} = (1, 2, 0)^T$.

Scheme of the Criss-Cross algorithm for sufficient LCP



Diagonal pivot



Exchange pivot

Input:

Problem LCP, where M is sufficient, $\bar{M} := -M$, $\bar{q} := q$, $r := 1$,

Begin

$J := \{\alpha \in I : \bar{q}_\alpha < 0\}$

While $(J \neq \emptyset)$ **do**

Select entering variable k

If $(\bar{m}_{kk} < 0)$ **then**

diagonal pivot on $\bar{m}_{kk}$

Else

$K := \{\alpha \in I : \bar{m}_{k\alpha} < 0\}$

If $(K = \emptyset)$ **then**

Stop: $\mathcal{F}_\oplus = \emptyset$

LCP problem has no feasible solution.

Else

Select exchange variable l

exchange pivot on $\bar{m}_{kl}$ and m_{lk}

Endif

Endif

Endwhile

Stop: $\mathcal{F}^* \neq \emptyset$

A feasible complementary solution has been computed.

End

Variants of the finite Criss-Cross algorithm for LCP

Authors & year	pivot rule	matrix class
Klafszky, Terlaky (1992)	minimal index	bisymmetric
Väliaho (1992)	minimal index	bisymmetric
den Hertog, Roos, Terlaky (1993)	minimal index	sufficient
Fukuda, Namiki, Tamura (1998)	minimal index	general
Akkeleș, Balogh, Illés (2004)	LIFO, MOSV	bisymmetric
Csizmadia, Illés (2006)	LIFO, MOSV	general
Csizmadia, Illés, Nagy a. (2007, 2013)	s-monotone	general

Lemma Csizmadia, Illés, (2007)

Minimal index-, LIFO- and MOSV pivot rules are **s***-monotone pivot rules.*

Theorem Csizmadia, Illés, Nagy A. (2007, 2013)

Criss-cross algorithm with **s***-monotone pivot rule is finite for the linear complementarity problem with sufficient (bisymmetric) matrices.*

From this theorem follows Fukuda–Terlaky (1992) duality theorem for sufficient LCP and generalize the results and method of den Hertog, Roos, Terlaky (1993).

General LCP - lack of sufficiency

- The sign structure of sufficient matrices is relatively easy to check (in case of pivot algorithms), $O(n^2)$ extra memory and $O(n)$ extra operations are needed when we are using criss-cross algorithm.
- Cycling of pivot algorithm is much harder to detect. For controlling the finiteness of the algorithm, to avoid possible cycling, we should (i) check the sign structures, (ii) save rows or columns of the active moving variable, and (iii) check the scalar products of the row (column) of the variables moved during the last pivot.

$$\begin{aligned} -u_2 + v_1 &= 1 \\ -u_1 + v_2 &= -1 \\ u_1, u_2, v_1, v_2 &\geq 0 \\ u_1 v_1 &= 0 \\ u_2 v_2 &= 0 \end{aligned}$$

0	−1	1
−1	0	−1
0	−1	−1
−1	0	1

Basic solution $v_1 = 1,\ v_2 = -1$ and $u_1 = u_2 = 0$. Exchange pivot: v_2 leaves, u_1 enters and v_1 leaves, u_2 enters. Then the solution: $u_1 = 1,$ $u_2 = -1$ and $v_1 = v_2 = 0$.

Again exchange pivot: u_2 leaves, v_1 enters and u_1 leaves, v_1 enters. We got back the starting pivot tableau. **Cycling ! The matrix M is not sufficient (P_0) !**

Scheme of the Criss-Cross algorithm for general LCP

Extra memory and processor use

1. $O(n^2)$ extra memory
2. $O(n)$ extra operations

Controlling cycling

1. Check sign structures
2. Save rows or columns of the moving leading variable
3. Check products with last movement's row or column variable pair

Results

1. Criss-cross algorithm with **s**-monotone pivot rules for general LCP is finite.
2. The EP-theorem of Fukuda, Namiki, Tamura (1998) can be derived from our result.
3. Numerical experiments show that LIFO- and MOSV pivot rules are more efficient for solving practical problems then the minimal index pivot rule.

Input: Initial basic tableau, $r = 1$, initalaze Q.
Begin
  While $((\mathcal{J} := \{i \in: \bar{q}_i < 0\}) \neq \emptyset)$ **do**
    $\mathcal{J}_{\max} := \{\beta \in: \mathbf{s}(\beta) \geq \mathbf{s}(\alpha)$, for all $\alpha \in \mathcal{J}\}$.
    Let $k \in \mathcal{J}_{\max}$ be arbitrary.
    Check $-\mathbf{u}' \cdot \mathbf{v}'' - \mathbf{u}'' \cdot \mathbf{v}''$ with the help of $Q(k)$.
    If $(-\mathbf{u}' \cdot \mathbf{v}'' - \mathbf{u}'' \cdot \mathbf{v}'' \lneqq 0)$ **then**
      Stop: M is not sufficient, certificate: $\mathbf{u}' - \mathbf{u}''$.
    Endif
    If $(t_{kk} < 0)$ **then**
      diagonal pivot on t_{kk}, update **s**.
      $Q(k) = [\mathcal{J}_B, \bar{\mathbf{t}}_q]$, $r := r + 1$.
    ElseIf $(t_{kk} > 0)$
      Stop: M is not sufficient, create certificate.
    Else /* $t_{kk} = 0$ */
      $K := \{\alpha \in I : \bar{t}_{k\alpha} < 0\}$
      If $(K = \emptyset)$ **then**
        Stop: (D-LCP) solution.
      Else
        $_{\max} = \{\beta \in K : \mathbf{s}(\beta) \geq \mathbf{s}(\alpha)$, for all $\alpha \in K\}$.
        Let $l \in_{\max}$ be arbitrary.
        If $((\mathbf{t}_k, \mathbf{t}^k)$ or $(\mathbf{t}_l, \mathbf{t}^l)$ sign structure is violated) **then**
          Stop: M is not sufficient, create certificate.
        Endif
        Exchange pivot on t_{kl} and t_{lk}, update **s** first for (u_k, v_k), then for (u_l, v_l) as in a next iteration.
        $Q(k) = [J_B, \bar{\mathbf{t}}_q]$, $Q(l) = [\emptyset, \mathbf{0}]$, $r := r + 2$.
      Endif
    Endif
  EndWhile
  Stop: we have a complementary feasible solution.
End

Central path problem

Sonnevend 1985; Megiddo 1989

$$-M\,\mathbf{u} + \mathbf{v} = \mathbf{q}, \quad \mathbf{u} > \mathbf{0},\ \mathbf{v} > \mathbf{0}, \quad \mathbf{u}\,\mathbf{v} = \mu\,\mathbf{e} \qquad \mu > 0$$

Theorem Kojima, Megiddo, Noma, Yoshise (1991)

Let M be a $P_*(\kappa)$–matrix. Then following statements are equivalent

- $\mathcal{F}_+ \neq \emptyset$
- $\forall \mathbf{w} > \mathbf{0}, \exists!\,(\mathbf{x}, \mathbf{s}) \in \mathcal{F}_+ : \mathbf{x}\,\mathbf{s} = \mathbf{w}$,
- $\forall \mu > 0,\ \exists!\,(\mathbf{x}, \mathbf{s}) \in \mathcal{F}_+ : \mathbf{x}\,\mathbf{s} = \mu\,\mathbf{e}$.

Proposition

If $M \in \mathbb{R}^{n \times n}$ is a $\mathcal{P}_*(\kappa)$–matrix (P_0–matrix) then

$$M' = \begin{bmatrix} -M & I \\ S & X \end{bmatrix} \text{ is a nonsingular matrix}$$

for any positive diagonal matrices $X, S \in \mathbb{R}^{n \times n}$.

Newton-system, scaling, proximity measure

Let $(\mathbf{x},\mathbf{s}) \in \mathcal{F}^+$ be given. Find (approximate) the unique solution $(\hat{\mathbf{x}},\hat{\mathbf{s}}) \in \mathfrak{C} \subset \mathcal{F}^+$ of central path problem with respect to μ in form $\hat{\mathbf{x}} = \mathbf{x} + \Delta x, \;\; \hat{\mathbf{s}} = \mathbf{s} + \Delta s.$

Newton–system:

$$\begin{aligned} -\,M\,\Delta x + \quad \Delta s &= \mathbf{0} \\ \mathbf{s}\,\Delta x + \mathbf{x}\,\Delta s &= \mu\,\mathbf{e} - \mathbf{x}\,\mathbf{s} \end{aligned}$$

Let us define the following vectors

$$\mathbf{v} = \sqrt{\frac{\mathbf{x}\,\mathbf{s}}{\mu}}, \quad \mathbf{d} = \sqrt{\frac{\mathbf{x}}{\mathbf{s}}}, \quad \mathbf{d}_x = \frac{\mathbf{d}^{-1}\Delta x}{\sqrt{\mu}} = \frac{\mathbf{v}\,\Delta x}{\mathbf{x}}, \quad \mathbf{d}_s = \frac{\mathbf{d}\,\Delta s}{\sqrt{\mu}} = \frac{\mathbf{v}\Delta s}{\mathbf{s}}$$

Let $\bar{M} = DMD$ then the rescaled Newton–system:

$$\begin{aligned} -\bar{M}\,\mathbf{d}_x + \mathbf{d}_s &= \mathbf{0} \\ \mathbf{d}_x + \mathbf{d}_s &= \mathbf{v}^{-1} - \mathbf{v} \end{aligned}$$

proximity measure: $\quad \psi(\mathbf{x},\mathbf{s},\mu) = \psi(\mathbf{v}) := \|\mathbf{v}^{-1} - \mathbf{v}\|$

Variants of IPAs for LCP

Authors & year	algorithm type	matrix class
Dikin (1967)	affine scaling	SS, SPD
Sonnevend (1985)	path following	SS, SPD
Kojima, Mizuno, Yoshise (1989)	logarithmic barrier	SS, PSD
Kojima, Megiddo, Noma, Yoshise (1991)	logarithmic barrier	$P*(\kappa)$
⋮		
Ji, Potra, Sheng (1995)	predictor-corrector	$P*(\kappa)$
Illés, Roos, Terlaky (1997)	affine scaling	$P*(\kappa)$
Potra (2002)	MTY-pc	$P*(\kappa)$
Potra, Lin (2005)	MTY-pc	sufficient
Illés, M. Nagy (2007)	MTY-pc	$P*(\kappa)$
Illés, M. Nagy, Terlaky (2009)	affine scaling	general
Illés, M. Nagy, Terlaky (2008)	logarithmic barrier	general
Illés, M. Nagy, Terlaky (2009)	MTY-pc	general
Illés, M. Nagy (2009)	adaptive, path following	general

Long step algorithm – LP

$$\delta_c(\mathbf{xs}, \mu) := \left\| \sqrt{\frac{\mathbf{xs}}{\mu}} - \sqrt{\frac{\mu}{\mathbf{xs}}} \right\|$$

$\mathbf{x} := \mathbf{x}^0$, $\mathbf{s} := \mathbf{s}^0$;
while $\mathbf{x}^T\mathbf{s} \geq \varepsilon$ **do**
  $\mu = (1-\gamma)\mu$;
  while $\delta_c(\mathbf{xs}, \mu) \geq \tau$ **do**
    compute the Newton direction $(\Delta\mathbf{x}, \Delta\mathbf{s})$;
    $\bar{\alpha} = \operatorname{argmin}\{\delta_c(\mathbf{x}(\alpha)\mathbf{s}(\alpha), \mu) : \ (\mathbf{x}(\alpha), \mathbf{s}(\alpha)) > \mathbf{0}\}$;
    $\mathbf{x} = \mathbf{x}(\bar{\alpha}), \ \mathbf{s} = \mathbf{s}(\bar{\alpha})$;
  end
end

Long step algorithm – $\mathcal{P}_*$ (Potra's idea)

$$\delta_c(\mathbf{xs}, \mu) := \left\| \sqrt{\frac{\mathbf{xs}}{\mu}} - \sqrt{\frac{\mu}{\mathbf{xs}}} \right\|$$

$\mathbf{x} := \mathbf{x}^0$, $\mathbf{s} := \mathbf{s}^0$, $\kappa := 1$;
while $\mathbf{x}^T\mathbf{s} \geq \varepsilon$ **do**
  $\mu = (1-\gamma)\mu$;
  while $\delta_c(\mathbf{xs}, \mu) \geq \tau$ **do**
    compute the Newton direction $(\Delta\mathbf{x}, \Delta\mathbf{s})$;
    if (the Newton direction does not exist or not unique) **then**
      return the matrix is not $\mathcal{P}_0$;
    $\bar{\alpha} = \text{argmin}\left\{\delta_c(\mathbf{x}(\alpha)\mathbf{s}(\alpha), \mu) : \ (\mathbf{x}(\alpha), \mathbf{s}(\alpha)) > \mathbf{0}\right\};$
    if $\left(\delta_c^2(\mathbf{xs}, \mu) - \delta_c^2(\mathbf{x}(\bar{\alpha})\mathbf{s}(\bar{\alpha}), \mu) < \frac{5}{3(1+4\kappa)}\right)$ **then**
      determine $\kappa(\Delta\mathbf{x}, \Delta\mathbf{s})$;
      if($\kappa(\Delta\mathbf{x}, \Delta\mathbf{s})$ is not defined) **then**
        return the matrix is not $\mathcal{P}_*$;
      else $\kappa = 2\kappa$;
    else $\mathbf{x} = \mathbf{x}(\bar{\alpha})$, $\mathbf{s} = \mathbf{s}(\bar{\alpha})$;
  end
end

Long step algorithm – $\mathcal{P}_*$

$$\delta_c(\mathbf{xs},\mu) := \left\| \sqrt{\frac{\mathbf{xs}}{\mu}} - \sqrt{\frac{\mu}{\mathbf{xs}}} \right\|$$

$\mathbf{x} := \mathbf{x}^0$, $\mathbf{s} := \mathbf{s}^0$, $\kappa := 0$;
while $\mathbf{x}^T\mathbf{s} \geq \varepsilon$ **do**
  $\mu = (1-\gamma)\mu$;
  while $\delta_c(\mathbf{xs},\mu) \geq \tau$ **do**
    compute the Newton direction $(\Delta\mathbf{x}, \Delta\mathbf{s})$;
    if (the Newton direction does not exist or not unique) **then**
      return the matrix is not $\mathcal{P}_0$;
    $\bar{\alpha} = \operatorname{argmin}\left\{\delta_c(\mathbf{x}(\alpha)\mathbf{s}(\alpha),\mu) : \ (\mathbf{x}(\alpha),\mathbf{s}(\alpha)) > \mathbf{0}\right\};$
    if $\left(\delta_c^2(\mathbf{xs},\mu) - \delta_c^2(\mathbf{x}(\bar{\alpha})\mathbf{s}(\bar{\alpha}),\mu) < \frac{5}{3(1+4\kappa)}\right)$ **then**
      determine $\kappa(\Delta\mathbf{x}, \Delta\mathbf{s})$;
      if ($\kappa(\Delta\mathbf{x}, \Delta\mathbf{s})$ is not defined) **then**
        return the matrix is not $\mathcal{P}_*$;
      else $\kappa = \kappa(\Delta\mathbf{x}, \Delta\mathbf{s})$;
    else $\mathbf{x} = \mathbf{x}(\bar{\alpha}),\ \mathbf{s} = \mathbf{s}(\bar{\alpha})$;
  end
end

$$\kappa(\Delta\mathbf{x},\Delta\mathbf{s}) := -\frac{1}{4}\frac{\Delta\mathbf{x}^T\Delta\mathbf{s}}{\sum_{I_+}\Delta x_i \Delta s_i}$$

Long step algorithm – general matrix

$$\delta_c(\mathbf{xs}, \mu) := \left\| \sqrt{\frac{\mathbf{xs}}{\mu}} - \sqrt{\frac{\mu}{\mathbf{xs}}} \right\|$$

x $:= \mathbf{x}^0$, **s** $:= \mathbf{s}^0$, $\kappa := 0$;
while $\mathbf{x}^T\mathbf{s} \geq \varepsilon$ **do**
  $\mu = (1-\gamma)\mu$;
  while $\delta_c(\mathbf{xs}, \mu) \geq \tau$ **do**
    compute the Newton direction $(\Delta\mathbf{x}, \Delta\mathbf{s})$;
    if (the Newton direction does not exist or not unique) **then**
      return the matrix is not $\mathcal{P}_0$;
    $\bar{\alpha} = \operatorname{argmin}\{\delta_c(\mathbf{x}(\alpha)\mathbf{s}(\alpha), \mu) : \ (\mathbf{x}(\alpha), \mathbf{s}(\alpha)) > \mathbf{0}\}$;
    if $\left(\delta_c^2(\mathbf{xs}, \mu) - \delta_c^2(\mathbf{x}(\bar{\alpha})\mathbf{s}(\bar{\alpha}), \mu) < \frac{5}{3(1+4\kappa)}\right)$ **then**
      determine $\kappa(\Delta\mathbf{x}, \Delta\mathbf{s})$;
      if($\kappa(\Delta\mathbf{x}, \Delta\mathbf{s})$ is not defined) **then**
        return the matrix is not $\mathcal{P}_*$;
      else $\kappa = \kappa(\Delta\mathbf{x}, \Delta\mathbf{s})$;
    else $\mathbf{x} = \mathbf{x}(\bar{\alpha})$, $\mathbf{s} = \mathbf{s}(\bar{\alpha})$;
  end
end

$$\kappa(\Delta\mathbf{x}, \Delta\mathbf{s}) := -\frac{1}{4}\frac{\Delta\mathbf{x}^T\Delta\mathbf{s}}{\sum_{I_+} \Delta x_i \Delta s_i}$$

Long step algorithm – general matrix, polynomial (approximation) algorithm with $\tilde{\kappa} > 0$ parameter

$$\delta_c(\mathbf{xs}, \mu) := \left\| \sqrt{\frac{\mathbf{xs}}{\mu}} - \sqrt{\frac{\mu}{\mathbf{xs}}} \right\|$$

$\mathbf{x} := \mathbf{x}^0$, $\mathbf{s} := \mathbf{s}^0$, $\kappa := 0$;
while $\mathbf{x}^T\mathbf{s} \geq \varepsilon$ **do**
  $\mu = (1-\gamma)\mu$;
  while $\delta_c(\mathbf{xs}, \mu) \geq \tau$ **do**
    compute the Newton direction $(\Delta\mathbf{x}, \Delta\mathbf{s})$;
    if (the Newton direction does not exist or not unique) **then**
      return the matrix is not $\mathcal{P}_0$;
    $\bar{\alpha} = \operatorname{argmin}\{\delta_c(\mathbf{x}(\alpha)\mathbf{s}(\alpha), \mu) : \ (\mathbf{x}(\alpha), \mathbf{s}(\alpha)) > \mathbf{0}\}$;
    if $\left(\delta_c^2(\mathbf{xs}, \mu) - \delta_c^2(\mathbf{x}(\bar{\alpha})\mathbf{s}(\bar{\alpha}), \mu) < \frac{5}{3(1+4\kappa)}\right)$ **then**
      determine $\kappa(\Delta\mathbf{x}, \Delta\mathbf{s})$;
      if($\kappa(\Delta\mathbf{x}, \Delta\mathbf{s})$ is not defined or $\kappa > \tilde{\kappa}$) **then**
        return the matrix is not $\mathcal{P}_*(\tilde{\kappa})$;
      else $\kappa = \kappa(\Delta\mathbf{x}, \Delta\mathbf{s})$;
    else $\mathbf{x} = \mathbf{x}(\bar{\alpha})$, $\mathbf{s} = \mathbf{s}(\bar{\alpha})$;
  end
end

$$\kappa(\Delta\mathbf{x}, \Delta\mathbf{s}) := -\frac{1}{4}\frac{\Delta\mathbf{x}^T\Delta\mathbf{s}}{\sum_{I_+} \Delta x_i \Delta s_i}$$

Solving the corresponding LCP

Standard Lemke's algorithm (1968) may not work, (Dang, Ye, Zhu, 2008). Gives the trivial solution $\mathbf{u} = \mathbf{0}$ and $\mathbf{v} = \mathbf{e}$. To exclude the trivial solution, we shall rewrite the problem in an equivalent (homogeneous) form

$$A^T\mathbf{u} - \mathbf{e}\,\xi + \mathbf{v} = \mathbf{0},\ -\mathbf{e}^T\mathbf{u} = -1,\ (\mathbf{u}, \xi, \mathbf{v}, \zeta) \geq \mathbf{0},\ \mathbf{u}^T\mathbf{v} + \xi\,\zeta = 0.$$

The standard Lemke's algorithm stops in the second iteration with secondary ray.

The previous problem always has interior feasible solution, so the equivalent problem as well.

$\rightsquigarrow$ Apply **interior point algorithm** or **criss-cross algorithm**?

Computational results: test problems, ...

- The Arrow – Debreu model always has a solution [Arrow – Debreu Theorem (1954)]. When Leontief utility function is used then it is enough to compute a non-trivial solution of the $(LCP_{AD-pc-Luf})$ [Ye (2007)].
- The matrix of the $(LCP_{AD-pc-Luf})$ usually is not a *sufficient matrix*, but it is easy to generate initial, starting, feasible point.
- We have randomly generated 10 sparse matrices of size $n \times n$, where $n = 10, 20, 40, 60, 80, 100, 200$. The entries of the matrices where chosen from the $[0, 1]$ interval.
- We have tested our generalized *long-step path following* - and *predictor-corrector interior point algorithms* starting from 100 randomly generated feasible solution of the problem.
- Our generalized IPAs stops in polynomial time (depending on $\tilde{\kappa}$) (i) with a solution of the (LCP) or (ii) with a certificate that the matrix does not belong to the class of $P_*(\tilde{\kappa})$. [Solution of the problem in EP-sense.]

Computational results: long–step path following IPA

10 different, randomly generated matrices for each size, 100 randomly generated starting feasible point for each matrix. Average of 1000 runs for each size.
Special parameters of the algorithm: $\tilde{\kappa} = 100$, $Bound - it = 1000$ and $Bound - line - search = 20$.

n	mean-T	mean-it	max-T	max-it	mean-supp	# sol	# diff sol
10	0.010645	36.984	0.391	47	4.047	86.3	9.3
20	0.024352	36.040	0.984	48	9.159	73.5	15.1
40	0.074115	31.820	4.016	49	24.707	47.3	20.8
60	0.168232	28.790	12.532	50	44.112	31.6	17.6
80	0.390925	26.924	28.516	50	63.709	23.8	15.5
100	0.505757	25.062	13.062	50	86.314	15.8	12.5
200	2.825490	22.427	141.703	50	197.741	1.3	1.3

Results of the long–step path following IPA [developed in MATLAB and run on PC (1.8GHz)] for the (LCP) corresponding to Arrow – Debreu model with Leontief utility function

Computational results: predictor–corrector IPA

10 different, randomly generated matrices for each size, 100 randomly generated starting feasible point for each matrix. Average of 1000 runs for each size.

Special parameters of the algorithm: $\tilde{\kappa} = 100$ and $Bound - it = 1000$.

n	mean-T	mean-it	max-T	max-it	mean-supp	# sol	# diff sol
10	0.007468	13.610	0.094	139	4.082	84.6	6.0
20	0.015384	16.996	0.172	197	9.201	71.8	14.1
40	0.056163	22.725	1.812	745	23.522	49.8	19.5
60	0.136329	27.707	1.156	237	41.479	36.4	20.4
80	0.315274	32.533	7.609	779	61.803	26.2	15.8
100	0.557608	31.241	6.984	369	85.301	16.7	11.7
200	5.632738	56.599	99.500	1000	193.772	3.5	3.5

Results of the Predictor–corrector IPA [developed in MATLAB and run on PC (1.8GHz)] for the (LCP) corresponding to Arrow – Debreu model with Leontief utility function

Thank you for your attention

A. A. Akkeleş, L. Balogh, T. Illés, *New variants of the criss-cross method for linearly constrained convex quadratic programming*, European Journal of Operational Research, Vol. 157, No. 1: 74-86, (2004).

K. J. Arrow, G. Debreu, *Existence of an equilibrium for competitive economy*, Econometrica 22:265-290 (1954).

K. Cameron, J. Edmonds, *Existentially polytime theorems,* in: W. Cook, P.D. Seymour (Editors), Polyhedral Combinatorics, DIMACS Series in Discrete Mathematics and Theoretical Computer Science AMS pp. 83-100 (1990).

R.W. Cottle, J.-S. Pang, V. Venkateswaran, *Sufficient matrices and the linear complementarity problem*, Linear Algebra Applications, Vol. 114/115, pp. 230-249 (1989).

R.W. Cottle, J.-S. Pang, R. E. Stone, *The Linear Complementarity Problem*, Academic Press, Boston, 1992.

Zs. Csizmadia and T. Illés *New criss-cross type algorithms for linear complementarity problems with sufficient matrices*, Optimization Methods and Software, Vol. 21, pp. 247-266, (2006).

Zs. Csizmadia and T. Illés, *The s-Monoton Index Selection Rule for Linear Optimization Problems*, unpublished manuscript, 2007.

Zs. Csizmadia, *New pivot based methods in linear optimization, and an application in petroleum industry*, PhD Thesis, Department of Operations Research, Eötvös Loránd University of Sciences, May 2007, Budapest, Hungary.

C. Dang, Y. Ye, Z. Zhu, *An interior-point path-following algorithm for computing a Leontief economy equilibrium*, March 8, 2008, Working Paper, Department of Management Science and Engineering, Stanford University, Stanford, California, US.

Thank you for your attention

- I. I. Dikin, *Iterative solution of problems of linear and quadratic programming*, Soviet Mathematics Doklady, 8:674-675, 1967.
- K. Fukuda, T. Terlaky, *Linear complementarity and oriented matroids*, Journal of the Operations Research Society of Japan, Vol. 35, pp. 45-61 (1992).
- K. Fukuda, M. Namiki, A. Tamura, *EP theorems and linear complementarity problems*, Discrete Applied Mathematics, Vol. 84, pp. 107-119 (1998).
- K. Fukuda and T. Terlaky, *Criss-cross methods: A fresh view on pivot algorithms*, Mathematical Programming Series B, 79 (1997) 369–395.
- D. den Hertog, C. Roos, T. Terlaky, *The linear complementarity problem, sufficient matrices, and the criss-cross method*, Linear Algebra and Its Applications, Vol. 187, pp. 1-14 (1993).
- T. Illés, C. Roos, and T. Terlaky, *Polynomial Affine–Scaling Algorithms for $P_*(\kappa)$ Linear Complementarity Problems*, In P. Gritzmann, R. Horst, E. Sachs, R. Tichatschke, editors, *Recent Advances in Optimization*, Proceedings of the 8^{th} French-German Conference on Optimization, Trier, July 21-26, 1996, Lecture Notes in Economics and Mathematical Systems 452, pp. 119-137, Springer Verlag, 1997.
- T. Illés, J. Peng, C. Roos and T. Terlaky, *A Strongly Polynomial Rounding Procedure Yielding A Maximally Complementary Solution for $P*(\kappa)$ Linear Complementarity Problems*, SIAM Journal on Optimization, 11:320-340, 2000.
- T. Illés and K. Mészáros, *A New and Constructive Proof of Two Basic Results of Linear Programming*, Yugoslav Journal of Operations Research, 11:15-30, 2001.

Thank you for your attention

- T. Illés and T. Terlaky, *Pivot Versus Interior Point Methods: Pros and Cons*, European Journal of Operational Research, 140:6-26, 2002.
- T. Illés and M. Nagy, *A new variant of the Mizuno-Todd-Ye predictor-corrector algorithm for sufficient matrix linear complementarity problem*, European Journal of Operational Research 181 (2007) 1097-1111.
- T. Illés, M. Nagy and T. Terlaky, *EP theorem for dual linear complementarity problems*, Journal of Optimization Theory and Application, Vol. 140:233-238, 2009. (Electronic version available http://www.springerlink.com/content/w3x6401vx82631t2/)
- T. Illés, M. Nagy and T. Terlaky, *Polynomial Path-following Interior Point Algorithm for General Linear Complementarity Problems*, Journal of Global Optimization, in print, 2008. (Electronic version available http://www.springerlink.com/content/547557t2077t2525/)
- T. Illés, M. Nagy and T. Terlaky, *Polynomial Interior Point Algorithms for General Linear Complementarity Problems*, unpublished manuscript, 2008.
- J. Ji, F.A. Potra and R. Sheng, *A predictor–corrector method for the P_*–matrix LCP from infeasible starting points*, Optimization Methods and Software 6 (1995) 109-126.
- E. Klafszky, T. Terlaky, *The role of pivoting in proving some fundamental theorems of linear algebra*, Linear Algebra and Its Applications, Vol. 151, pp. 97-118 (1991).
- E. Klafszky and T. Terlaky, *Some generalization of the criss–cross method for quadratic programming*, Math. Oper. und Stat. Ser. Optimization 24 (1992) 127–139.

Thank you for your attention

M. Kojima, S. Mizuno and A. Yoshise, *A Polynomial-time Algorithm for a Class of Linear Complementarity Problems*, Mathematical Programming, 44 (1989) 1-26.

M. Kojima, N. Megiddo, T. Noma, A. Yoshise, *A unified approach to interior point algorithms for linear complementarity problems*, volume 538 of Lecture Notes in Computer Science. Springer Verlag, Berlin, Germany, (1991).

C. E. Lemke, *On complementary pivot theory*, in Dantzig and Veinott, eds. Mathematics of Decision Sciences, Part 1, American Mathematical Society, Providence, Rhode Island (1968), 95-114.

S. Mizuno, M. J. Todd and Y. Ye, *On Adaptive-step Primal-dual Interior-point Algorithms for Linear Programming*, Mathematics of Operations Research, Vol. 18, No. 4 964-981 (1993).

F.A. Potra, *The Mizuno-Todd-Ye algorithm in a Larger Neighborhood of the Central Path*, Europian Journal of Operational Research 143 257-267 (2002).

F.A. Potra, X. Liu, *Predictor-corrector methods for sufficient linear complementarity problems in a wide neighborhood of the central path*, Optim. Methods Softw., 20(1):145-168 (2005).

R.T. Rockafellar, *The elementary vectors of a subspace of* $\mathbb{R}^n$, in: R.C. Bose, T.A. Dowling (Eds.), Combinatorical Mathematics and Its Applications, Proceedings Chapel Hill Conference, pp. 104-127 (1969).

C. Roos, T. Terlaky, J.-Ph. Vial, *Theory and Algorithms for Linear Optimization, An Interior Point Approach*, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, New York, USA, (1997).

Thank you for your attention

- Gy. Sonnevend, *An "analytical center" for polyhedrons and new class of global algorithms for linear (smooth, convex) programming.* In *Lecture Notes in Control and Information Sciences 84*, pp. 866-876, New York 1985, Springer.
- Gy. Sonnevend, J. Stoer and G. Zhao, *On the Complexity of Following the Central Path of Linear Programs by Linear Extrapolation*, Methods of Operations Research 63 (1989) 19-31.
- T. Terlaky, *A convergent criss-cross method,* Math. Oper. und Stat. ser. Optimization, Vol. 16, No. 5, pp. 683-690 (1985).
- H. Väliaho, *A new proof of finiteness of the criss–cross method*, Math. Oper. und Stat. Ser. Optimization 25 (1992) 391–400.
- H. Väliaho, *P_* matrices are just sufficient*, Linear Algebra and Its Applications Vol. 239, pp. 103-108 (1996).
- L. Walras, *Elements of Pure Economics, or the Theory of Social Wealth*, 1874 (1899, 4th ed.; 1926, rev. ed., 1954, Engl. Transl.).
- Y. Ye, *Exchabge market equilibria with Leontief's utility: Freedom of pricing leads to rationality*, Theoretical Computer Science 378:134-142, (2007).
- Y. Ye, *A path to the Arrow-Debreu competitive market equilibrium*, Mathematical Programming, Ser. B 111:315-348, (2008).
- S. Zhang, *A new variant of criss–cross pivot algorithm for linear programming*, European Journal of Operation Research 116 (1999) 607-614.

Sufficient matrices

Definition

Let a vector $\mathbf{x} \in \mathbb{R}^{2n}$ be given. The vector $\mathbf{x}$ is strictly sign reversing if

$$\begin{aligned} x_i x_{\bar{\imath}} &\leq 0 \quad \text{for all indices } i = 1, \ldots, n \\ x_i x_{\bar{\imath}} &< 0 \quad \text{for at least one index } i \in \{1, \ldots, n\}, \end{aligned}$$

while it is strictly sign preserving if

$$\begin{aligned} x_i x_{\bar{\imath}} &\geq 0 \quad \text{for all indices } i = 1, \ldots, n \\ x_i x_{\bar{\imath}} &> 0 \quad \text{for at least one index } i \in \{1, \ldots, n\}. \end{aligned}$$

$$V := \left\{ (\mathbf{u}, \mathbf{v}) \in \mathbb{R}^{2n} \mid [-M, I](\mathbf{u}, \mathbf{v}) = \mathbf{0} \right\}$$

and

$$V^{\perp} := \left\{ (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2n} \mid [I, M^T](\mathbf{x}, \mathbf{y}) = \mathbf{0} \right\}.$$

Lemma

Fukuda, Namiki, Tamura (1998)

A matrix $M \in \mathbb{R}^{n \times n}$ is sufficient if and only if no strictly sign reversing vector exists in V, and no strictly sign preserving vector exists in $V^{\perp}$.

Sufficient matrices II.

A basis B of the linear system $-M\mathbf{u} + \mathbf{v} = \mathbf{q}$ is called complementary, if $\forall i \in \mathcal{I}$ exactly one of the columns corresponding to variables v_i and u_i is in the basis. Let $\{J_B, J_N\}$ be a (basic) partition of the index set $\mathcal{I}$. A short pivot tableau, $\bar{M} = [\bar{m}_{ij} : i \in J_B, j \in J_N]$, is called complementary, if the corresponding basis is complementary.

Lemma — Cottle, Pang, Venkateswaran, 1989

Let M be a sufficient matrix, B a complementary basis, $\bar{M}$ the corresponding short pivot tableau. Then

(a) $\bar{m}_{i\bar{\imath}} \geq 0$ *for all* $i \in J_B$*; furthermore*

(b) *for all* $i \in J_B$*, if* $\bar{m}_{i\bar{\imath}} = 0$ *then* $\bar{m}_{i\bar{\jmath}} = \bar{m}_{j\bar{\imath}} = 0$ *or* $\bar{m}_{i\bar{\jmath}} \cdot \bar{m}_{j\bar{\imath}} < 0$ *for all* $j \in J_B$, $j \neq i$.

By the permutation of $M \in \mathbb{R}^{n \times n}$ we mean the matrix $P^T M P$, where P is a permutation matrix.

Sufficient matrices III.

Lemma — den Hertog, Roos, Terlaky, 1993

Let $M \in \mathbb{R}^{n\times n}$ be a row (column) sufficient matrix. Then

(a) *any permutation of matrix M is row (column) sufficient,*

(b) *the product DMD is row (column) sufficient, where $D \in \mathbb{R}_+^{n\times n}$ is a positive diagonal matrix,*

(c) *every principal submatrix of M is row (column) sufficient.*

The matrix $\bar{M}$ is also sufficient after any number of arbitrary principal pivots, if M is sufficient.

For a matrix $M \in \mathbb{R}^{n\times n}$, and $\mathcal{J} \subseteq \{1,\ldots,n\}$, if $M_{\mathcal{J}\mathcal{J}}$ is nonsingular, the result of block pivot operation belonging to $\mathcal{J}$ is $M' = \eta(M,\mathcal{J})$.

Lemma — den Hertog, Roos, Terlaky, 1993

Let $M_{\mathcal{J}\mathcal{J}}$ be a nonsingular submatrix of the row (column) sufficient matrix M. Then $M' = \eta(M,\mathcal{J})$ is row (column) sufficient, where $|\mathcal{J}| = 2$.

The case when $1 < |\mathcal{J}| < n$ was proved by Väliaho in 1996.

Further properties of P_0–matrices

Corollary

Kojima, Megiddo, Noma, Yoshise (1991)

Let $M \in \mathbb{R}^{n\times n}$ be a $\mathcal{P}_0$-matrix, $\mathbf{x}, \mathbf{s} \in \mathbb{R}^n_+$. Then for all $\mathbf{a} \in \mathbb{R}^n$ the system

$$\begin{array}{rcrcl} -M\,\Delta\mathbf{x} & + & \Delta\mathbf{s} & = & \mathbf{0} \\ \mathbf{s}\Delta\mathbf{x} & + & \mathbf{x}\Delta\mathbf{s} & = & \mathbf{a} \end{array} \qquad (2)$$

has a unique solution $(\Delta\mathbf{x}, \Delta\mathbf{s})$.

Lemma

Potra (2002)

Let M be an arbitrary $n \times n$ real matrix and $(\Delta\mathbf{x}, \Delta\mathbf{s})$ be a solution of system (2). Then

$$\sum_{i\in\mathcal{I}_+} \Delta x_i \Delta s_i \leq \frac{1}{4}\left\|\frac{\mathbf{a}}{\sqrt{\mathbf{xs}}}\right\|^2 .$$

Properties of $\mathcal{P}_*(\kappa)$–matrices

Theorem Kojima, Megiddo, Noma, Yoshise (1991)

Let $M \in \mathbb{R}^{n\times n}$ be a matrix and $\kappa \in \mathbb{R}_\oplus$. The following statements are equivalent

- $M \in P_*(\kappa)$.
- *For every positive diagonal matrix D and every $\xi, \nu, h \in \mathbb{R}^n$, the relations*
$$D^{-1}\xi + D\eta = h,$$
$$-M\xi + \eta = 0$$
always imply
$$\xi^\top \eta \geqq -\kappa \|h\|_2^2$$
- *For every $\xi \in \mathbb{R}^n$ it is*
$$\xi^\top M\xi \geqq -\kappa \inf_D \|D^{-1}\xi + DM\xi\|_2^2,$$
where the infimum is taken over all positive diagonal matrices D.

Further properties of $\mathcal{P}_*(\kappa)$–matrices

Lemma — Illés, Roos, Terlaky (1997); Potra (2002)

Let matrix M be a $\mathcal{P}_*(\kappa)$–matrix and $\mathbf{x}, \mathbf{s} \in \mathbb{R}^n_+$, $\mathbf{a} \in \mathbb{R}^n$. Let $(\Delta\mathbf{x}, \Delta\mathbf{s})$ be the solution of (2). Then

$$\|\Delta\mathbf{x}\Delta\mathbf{s}\|_\infty \leq \left(\frac{1}{4} + \kappa\right) \left\| \frac{\mathbf{a}}{\sqrt{\mathbf{xs}}} \right\|^2, \quad \|\Delta\mathbf{x}\Delta\mathbf{s}\|_1 \leq \left(\frac{1}{2} + \kappa\right) \left\| \frac{\mathbf{a}}{\sqrt{\mathbf{xs}}} \right\|^2,$$

$$\|\Delta\mathbf{x}\Delta\mathbf{s}\|_2 \leq \sqrt{\left(\frac{1}{4} + \kappa\right)\left(\frac{1}{2} + \kappa\right)} \left\| \frac{\mathbf{a}}{\sqrt{\mathbf{xs}}} \right\|^2.$$

Lemma — Illés, Nagy M., Terlaky (2008-2009)

Let M be a real $n \times n$ matrix. If there exists a vector $\mathbf{x} \in \mathbb{R}^n$ such that $\kappa(\mathbf{x}) > \tilde{\kappa}$ $\;(\mathcal{I}_+(\mathbf{x}) = \{i \in \mathcal{I} : x_i(Mx)_i > 0\} = \emptyset)$, then the matrix M is not $\mathcal{P}_*(\tilde{\kappa})$ $(\mathcal{P}_*)$ and $\mathbf{x}$ is a certificate for this fact.

Some useful lemmas

Illés, Nagy M., Terlaky (2008-2009)

Lemma

If one of the following statements holds then the matrix M is not a $\mathcal{P}_*(\kappa)$-matrix.

1. *There exists a vector* $\mathbf{y} \in \mathbb{R}^n$ *such that*

$$(1+4\kappa) \sum_{i \in \mathcal{I}_+(\mathbf{y})} y_i w_i + \sum_{i \in \mathcal{I}_-(\mathbf{y})} y_i w_i < 0,$$

   where $\mathbf{w} = M\mathbf{y}$.

2. *There exists a solution* $(\Delta\mathbf{x}, \Delta\mathbf{s})$ *of the system (2) such that*

$$\max \left(\sum_{i \in \mathcal{I}_+} \Delta x_i \Delta s_i,\ -\sum_{i \in \mathcal{I}_-} \Delta x_i \Delta s_i \right) > \frac{1+4\kappa}{4} \left\| \frac{\mathbf{a}}{\sqrt{\mathbf{xs}}} \right\|^2, \qquad or$$

$$\|\Delta\mathbf{x}\Delta\mathbf{s}\|_\infty > \frac{1+4\kappa}{4} \left\| \frac{\mathbf{a}}{\sqrt{\mathbf{xs}}} \right\|^2 \qquad or \qquad \Delta x^T \Delta s < -\kappa \left\| \frac{\mathbf{a}}{\sqrt{\mathbf{xs}}} \right\|^2.$$

Complexity issues:

Illés, Nagy M., Terlaky (2008-2009)

Lemma

If after an inner iteration the decrease of the proximity is not sufficient, i.e., $\delta^2(\mathbf{xs},\mu) - \delta^2(\mathbf{x}(\bar\theta)\mathbf{s}(\bar\theta),\mu) < \frac{5}{3(1+4\kappa)}$*, then the matrix of the LCP is not* $\mathcal{P}_*(\kappa)$ *with the actual* κ *value, and the Newton direction* $\Delta\mathbf{x}$ *is a certificate for this fact.*

Lemma

At each iteration when the value of κ *is updated, then the new value of* κ *satisfies the inequality* $\delta^2(\mathbf{xs},\mu) - \delta^2(\mathbf{x}(\bar\theta)\mathbf{s}(\bar\theta),\mu) \geq \frac{5}{3(1+4\kappa)}$.

Theorem

Let $\tau = 2$, $\gamma = 1/2$ *and* $(\mathbf{x}^0, \mathbf{s}^0)$ *be a feasible interior point such that* $\delta_c(\mathbf{x}^0\mathbf{s}^0, \mu^0) \leq \tau$*. Then after at most* $\mathcal{O}\left((1+4\hat\kappa)n\log\frac{(\mathbf{x^0})^T\mathbf{s^0}}{\varepsilon}\right)$ *steps, where* $\hat\kappa \leq \tilde\kappa$ *is the largest value of parameter* κ *throughout the algorithm, the long-step path-following interior point algorithm either produces a point* $(\hat{\mathbf{x}}, \hat{\mathbf{s}})$ *such that* $\hat{\mathbf{x}}^T\hat{\mathbf{s}} \leq \varepsilon$ *and* $\delta_c(\hat{\mathbf{x}}\hat{\mathbf{s}}, \hat\mu) \leq \tau$ *or it gives a certificate that the matrix of the LCP is not* $\mathcal{P}_*(\tilde\kappa)$.

Embedding – starting point

Lemma Kojima, Megiddo, Noma, Yoshise, 1991

Let M be a real matrix. Then $M'=\begin{pmatrix} M & I \\ -I & O \end{pmatrix}$ is a $\mathcal{P}_0$, CS, $\mathcal{P}_*$, $\mathcal{P}_*(\kappa)$, PSD or SS matrix if and only if M belongs to the same matrix class.

(LCP') : $\qquad -M'\mathbf{x}'+\mathbf{s}'=\mathbf{q}' \quad \mathbf{x}',\mathbf{s}'\geq \mathbf{0} \quad \mathbf{x}'\,\mathbf{s}'=\mathbf{0},$ where

$$\mathbf{x}'=\begin{pmatrix}\mathbf{x}\\ \tilde{\mathbf{x}}\end{pmatrix},\ \mathbf{s}'=\begin{pmatrix}\mathbf{s}\\ \tilde{\mathbf{s}}\end{pmatrix},\ \mathbf{q}'=\begin{pmatrix}\mathbf{q}\\ \tilde{\mathbf{q}}\end{pmatrix},\ M'=\begin{pmatrix} M & I \\ -I & O \end{pmatrix}$$

Let $L\in\mathbb{R}_+$ such that $\tilde{\mathbf{q}}=\dfrac{2^{L+1}}{n^2}\,\mathbf{e}>\mathbf{x}$ for all $(\mathbf{x},\mathbf{s})$ feasible basic solution of the system $-M\mathbf{x}+\mathbf{s}=\mathbf{q},\ (\mathbf{x},\mathbf{s})\geq\mathbf{0}$. Appropriate value for L is

$L=\sum_{i=1}^{n}\sum_{j=1}^{n}\log_2(|m_{ij}|+1)+\sum_{i=1}^{n}\log_2(|q_i|+1)+2\log_2 n,$ which is an upper bound on the bit length of the problem

Starting interior point: $\quad \mathbf{x}=\dfrac{2^L}{n^2}\,\mathbf{e},\quad \tilde{\mathbf{x}}=\dfrac{2^{2L}}{n^3}\,\mathbf{e},\quad \mathbf{s}=\dfrac{2^L}{n^2}M\mathbf{e}+\dfrac{2^{2L}}{n^3}\,\mathbf{e}+\mathbf{q},\quad \tilde{\mathbf{s}}=\dfrac{2^L}{n^2}\,\mathbf{e}.$

Solution of original problem

Lemma

Let $(\mathbf{x}', \mathbf{s}') = (\mathbf{x}, \tilde{\mathbf{x}}, \mathbf{s}, \tilde{\mathbf{s}})$ be a solution of (LCP'). Then

1. *If $\tilde{\mathbf{x}} = \mathbf{0}$, $(\mathbf{x}, \mathbf{s})$ is a solution of the (LCP).*
2. *If M is column sufficient and $\tilde{\mathbf{x}} \neq \mathbf{0}$, the (LCP) has no solution.*

Solution of original problem:

- Embedding and try to solve the (LCP') with modified IPM
- if we get a solution $(\mathbf{x}, \tilde{\mathbf{x}}, \mathbf{s}, \tilde{\mathbf{s}})$
  $\tilde{\mathbf{x}} = \mathbf{0} \;\Rightarrow\; (\mathbf{x}, \mathbf{s})$ is a solution of $(P - LCP)$
  $\tilde{\mathbf{x}} \neq \mathbf{0} \;\Rightarrow\;$ *construct* $(\mathbf{u}, \mathbf{v})$
  $(\mathbf{u}, \mathbf{v})$ is a solution of $(D - LCP) \;\Rightarrow\; (P - LCP)$ has no solution
  $(\mathbf{u}, \mathbf{v})$ is not a solution of $(D - LCP) \;\Rightarrow\; M \notin \mathcal{P}_*, \quad /\mathbf{u}/$
- if we do not get solution
  Newton system is not regular $\Rightarrow M \notin \mathcal{P}_0, \quad /\mathbf{x}/$
  $\kappa(\Delta\mathbf{x}, \Delta\mathbf{s})$ is not defined $\Rightarrow M \notin \mathcal{P}_*, \quad /\Delta\mathbf{x}/$