LOEWNER'S THEOREM IN SEVERAL VARIABLES

MIKLÓS PÁLFIA

We provide characterizations of operator monotone and concave functions in several operator variables using LMIs and the theory of matrix convex sets. This completes the work of Agler-McCarthy-Young [2] providing characterizations restricted for commutative tuples of operators, hence to the several real variable situation, the work of Helton-McCullough-Vinnikov [6] characterizing free rational - thus already analytic - several variable matrix convex functions and the work of Pascoe-Tully-Doyle [15] characterizing free analytic matrix monotone functions in several variables.

For a free operator concave function we define its hypograph as the downward saturation of its graph with respect to the positive definite order. Then operator concavity of a free function is characterized by the matrix convexity of its hypograph. Given a closed matrix convex hypograph as a subset of a Cartesian product of the linear space of bounded linear operators, one can find its supporting linear functionals and represent them as linear pencils of operators on the tensor product of the linear space with its dual space. Then the linear pencil defines a linear matrix inequality (LMI) such that its extremal solution coincides with the value of the operator concave function. We establish an explicit solution formula for the extremal solutions of this LMI using the Schur complement. This LMI solution technique alone seems to have further applications to the general theory, in particular analytic rigidity, of matrix convex sets and LMIs.

The above approach leads to the extension of Loewner's classical representation theorem of operator concave and operator monotone functions from 1934, into the non-commutative several variable situation. Our theorem states that a free function defined on a k-variable free self-adjoint domain is operator monotone if and only if it has a free analytic continuation to the upper operator poly-halfspace $\Pi^k := \{X \in \mathcal{B}(E)^k : \Im X_i > 0, 1 \leq i \leq k\}$ for any Hilbert space E, mapping Π^k to Π. This approach also provides a new proof to the one-variable case of Loewner's theorem.

Date: August 11, 2016.

References

[1] Pick Interpolation for free holomorphic functions, American Journal of Mathematics 137:6 (2015), pp. 1685–1701

[2] J. Agler, J. E. McCarthy and N. Young, Operator monotone functions and Löwner functions of several variables, Ann. of Math., 176:3 (2012), pp. 1783–1826.

[3] W. Donoghue, Monotone Matrix Functions and Analytic Continuation, Springer, Berlin, Heidelberg, New York, (1974).

[4] F. Hansen and G.K. Pedersen, Jensen's inequality for operators and Löwner's theorem, Math. Ann., 258 (1982), pp. 229-241.

[5] J.W. Helton, "Positive" noncommutative polynomials are sums of squares, Ann. of Math. (2), 156:2 (2002), pp. 675-694.

[6] J.W. Helton, S.A. McCullough and V. Vinnikov, Noncommutative convexity arises from linear matrix inequalities, J. Func. Anal., Vol. 240, No. 1 (2006), pp. 105–191.

[7] J.W. Helton and S.A. McCullough, Every convex free basic semi-algebraic set has an LMI representation, Ann. Math., 176:2 (2012), pp. 979–1013.

[8] P. Kruszyński and S. L. Woronowicz, A non-commutative Gelfand-Naimark theorem, J. Operator Theory, 8 (1982), pp. 361-389.

[9] F. Kubo and T. Ando, Means of positive linear operators, Math. Ann., 246 (1980), pp. 205–224.

[10] Y. Lim and M. Pálfia, Matrix power means and the Karcher mean, J. Func. Anal., Vol. 262, No. 4 (2012), pp. 1498–1514.

[11] K. Löwner, Über monotone Matrixfunktionen, Math. Z., 38 (1934), pp. 177-216.

[12] M. Pálfia, Löwner's theorem in several variables, preprint (2016), http://arxiv.org/abs/1405.5076, 40 pages.

[13] M. Pálfia, Operator means of probability measures and generalized Karcher equations, Adv. Math. 289 (2016), pp. 951-1007.

[14] J. E. Pascoe and R. Tully-Doyle, Free Pick functions: representations, asymptotic behavior and matrix monotonicity in several noncommuting variables, preprint (2013), arXiv:1309.1791v2.

[15] G. Popescu, Noncommutative transforms and free pluriharmonic functions, Adv. Math., 220 (2009), pp. 831–893.

[16] B. Simon, Convexity, An Analytic Viewpoint, Cambridge University Press, Cambridge, UK, 2011.

[17] B. Simon, Loewner's Theorem on Monotone Matrix Functions, in preparation 2016.

[18] R. L. Schilling, R. Song, Z. Vondraček, Bernstein Functions: Theory and Applications, de Gruyter Studies in Mathematics 37, Springer, Berlin, 2010.

[19] D. S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov, Foundations of Free Noncommutative Function Theory, Mathematical Surveys and Monographs 199 (2014), 183 pp.

[20] E.P. Wigner and J.v. Neumann, Significance of Loewner's theorem in the quantum theory of collisions, Ann. of Math., 59 (1954), pp. 418-433.

Department of Mathematics, Sungkyunkwan University, Republic of Korea & Hungarian Academy of Sciences, Hungary

E-mail address: `palfia.miklos@aut.bme.hu`