An Improved Packing Lemma for Asynchronous Multiple Access Channel

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1) What is a "Packing Lemma"

The sharpest bounds for reliability functions can be obtained via the method of types. They rely upon packing lemmas that bound the number of codeword pairs/triplets/quadruples/etc with a given joint type.

2) Synchronous MAC

Packing lemma for Multiple Access Channel (MAC): there exist two constant composition codebooks $\mathcal{C}^X$, $\mathcal{C}^Y$ with rates R^X, R^Y such that the number of codeword quadruples with joint type $V = V_{X\hat{X}Y\hat{Y}}$ that is

$$\sum_{\boldsymbol{x},\hat{\boldsymbol{x}},\boldsymbol{y},\hat{\boldsymbol{y}}\in\mathcal{C}^X\times\mathcal{C}^X\times\mathcal{C}^Y\times\mathcal{C}^X} \mathbb{1}_{V_{X\hat{X}Y\hat{Y}}}(\boldsymbol{x},\hat{\boldsymbol{x}},\boldsymbol{y},\hat{\boldsymbol{y}})$$

is smaller than

$$\exp_2\left[2R^X + 2R^Y - I_V(X\wedge\hat{X}\wedge Y\wedge\hat{Y})\right].$$

where

$$\begin{aligned} I_V(X\wedge\hat{X}\wedge Y\wedge\hat{Y}) = \mathrm{H}(X)+\mathrm{H}(\hat{X})+ \\ +\mathrm{H}(Y)+\mathrm{H}(\hat{Y})- \\ -\mathrm{H}(X,\hat{X},Y,\hat{Y}) \end{aligned}$$

The proof is quite standard with random selection uniformly from type class.

4) AMAC with Single Codebook Pair

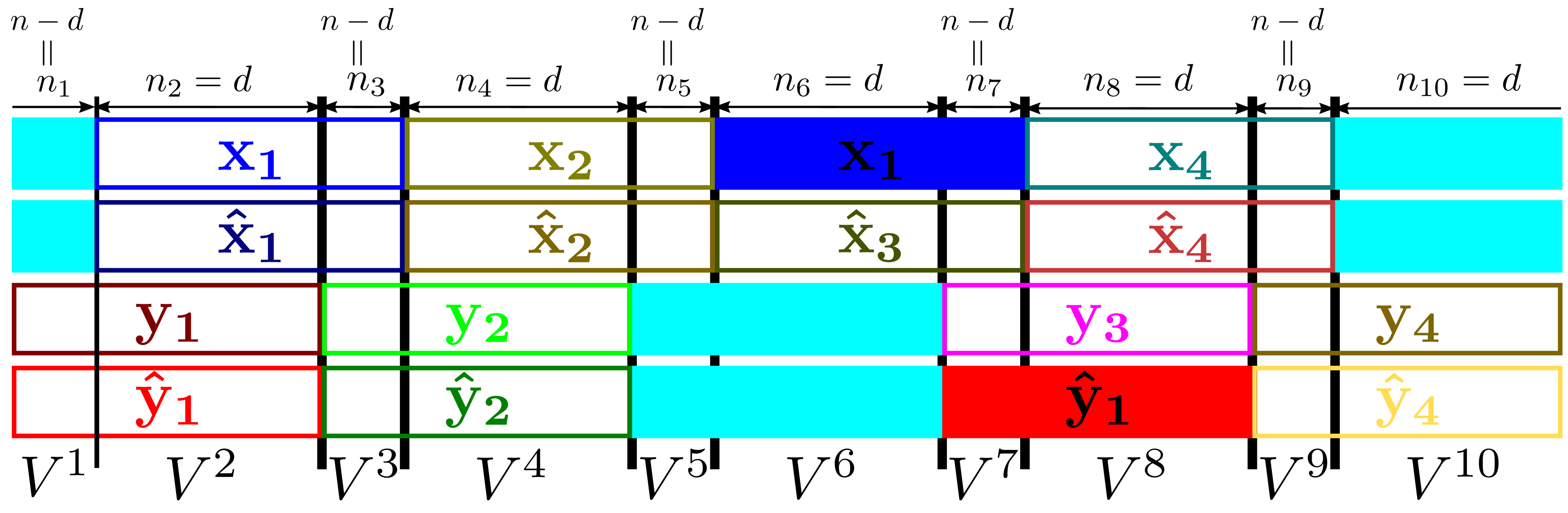

Figure 2: An expamle for codeword repetition

New Result: The Packing lemma and the asynchronous error exponent of [1] also holds for an appropriately chosen single codebook pair. The technique to deal with the possibility of some codewords appearing more than once is visualized by the example above. Namely let $N = 4$ and consider sequences where the codewords colored blue and red are identical (all others are different).

For this case the standard random coding technique gives:

$$\begin{aligned} p(n)\exp_2\Big[7nR^X + 7nR^Y - \\ -\sum_{i=1}^{6} n_i(\mathrm{I}_{V^i}(X\wedge\hat{X}\wedge Y\wedge\hat{Y}) - \\ -n_7\,\mathrm{I}_{V^7}\left(\hat{X}\wedge Y\wedge(X,\hat{Y})\right) - \\ -\sum_{i=8}^{10} n_i\,\mathrm{I}_{V^i}(X\wedge\hat{X}\wedge Y\wedge\hat{Y})\Big] \end{aligned} \qquad (1)$$

for the expected value of the number of quadruple of codeword sequences with this repetition pattern that have subtype sequence $V_1, V_2, \ldots, V_{10}$.

The corresponding bound without repetition would be

$$\begin{aligned} p(n)\exp_2\Big[8nR^X + 8nR^Y - \\ -\sum_{i=1}^{10} n_i(\mathrm{I}_{V^i}(X\wedge\hat{X}\wedge Y\wedge\hat{Y})\Big] \end{aligned} \qquad (2)$$

The difference of the two exponents is

$$nR^X + nR^Y - (n-d)\,\mathrm{I}_V(X\wedge\hat{Y}) \qquad (3)$$

If this is positive, then (2) upper bounds (1), thus, the bound originally derived with multiple codebooks still holds. We need to deal with the case when (3) is negative.

3) Asynchronous MAC (AMAC)

The codeword starting times of the two senders are different, **relative delay** d may be unknown to, or chosen by the senders. After N codewords, a **synchronization sequence** is sent. Sequences in a window of length $(N+1)n$ formed by consecutive codewords including the synch-sequence are partitioned into $2N+2$ **subblocks** (by the ends of the codewords) of length n_i equal to $n-d$ for i odd and d for i even (see Figure 1).

Upper bounds are needed for the number of quadruples of sequences with given **subtype sequence** $\boldsymbol{V} = (V^1, V^2, \ldots, V^N)$, where V^i is the joint type of the quadruples corresponding to the i-th subblock. These bounds should simultaneously hold for all possible d and $\boldsymbol{V}$.

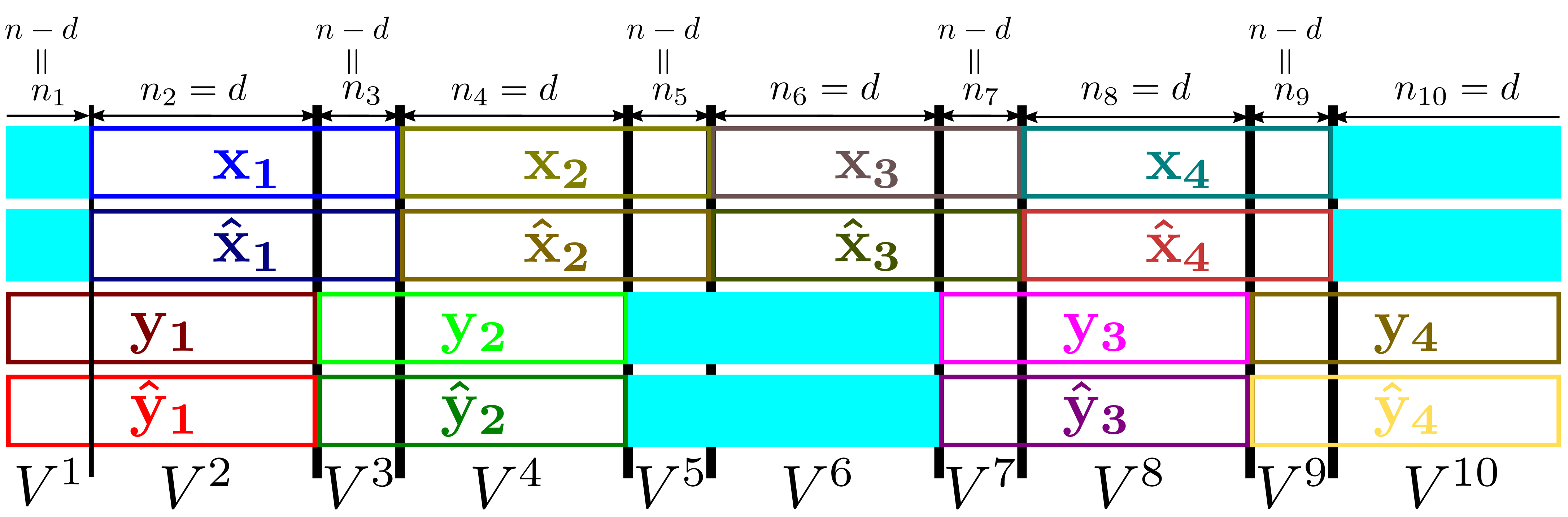

Figure 1: Visualizing the quadruple of codeword/subblock/subtype sequences for $N = 4$

In [1] such bound have been obtained when subsequent codewords were chosen from N suitable codebooks $C_1^X, C_2^X, \ldots, C_N^X$ resp. $C_1^Y, C_2^Y, \ldots, C_N^Y$. The latter rather than a single codebook pair C^X, C^Y has been needed for a technical reason. The random coding proof used independence of all codewords in the considered window which breaks down for sequences with some codeword appearance more than once.

The packing lemma in [1] has been used to give error exponent for AMAC. It has turned out that in certain cases, although the sender could be synchronized, **Controlled Asynchronism** may improve reliability.

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5) Solution

Packing lemma is extended to include bounds for auxiliary patterns. For the example above we need the pattern:

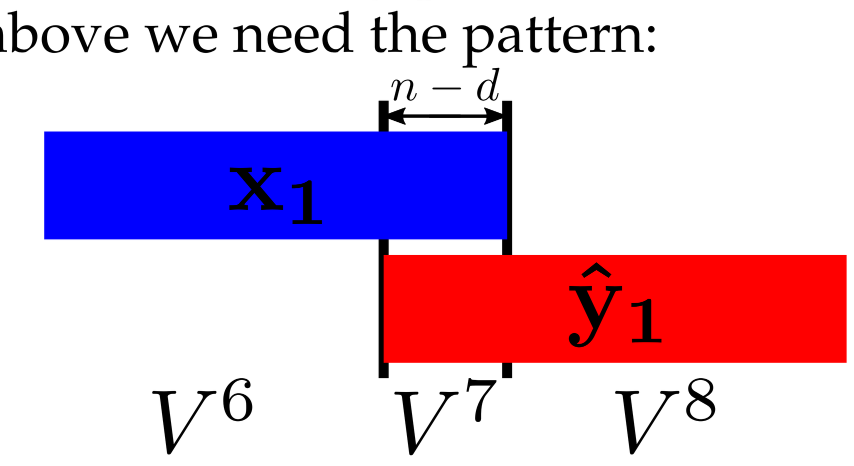

Figure 3: The auxiliary pattern

For randomly chosen codewords the expected value of the number of codeword pairs with subtype sequence $V_X^6, V_{X\hat{Y}}^7, V_{\hat{Y}}^8$ is bounded above by

$$p(n)\exp_2\left[nR^X + nR^Y - (n-d)\,\mathrm{I}_V(X\wedge\hat{Y})\right]$$

Hence, for some codebook pairs, in addition to (1) the bound above also holds. If (3) is negative, then the number of codeword pairs that have the joint type $V_{X\hat{Y}}^7$ is smaller than 1, thus equal to zero. So, for these subtype sequences the exponent in (2) also holds.

References

[1] L. Farkas and T. Kói, "Controlled asynchronism improves error exponent," in *Information Theory Proceedings (ISIT)*, Jun. 2015, pp. 2638–2642.