An equilibrium result on the circle

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Problem. **Linear polarization constant**. Let X be a normed space with $Y := X^\star$ its dual and S_X, S_Y the respective unit spheres. Find

$$C_n(X) := \inf_{y_1,\dots,y_n \in Y} \left\| \prod_{j=1}^{n} \langle y_j ; x \rangle \right\|_X = \inf_{y_1,\dots,y_n \in Y} \sup_{x \in S_X} \prod_{j=1}^{n} \left| \langle y_j ; x \rangle \right| .$$

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P. Kirwan – If $X = \mathbb{R}^2$ (with the Euclidean distance) we can rewrite the problem for the circle C:

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Ambrus, Ball and Erdélyi considered the following. Instead of products, i.e. geometric averages, consider harmonic means – if the conjectural inequality holds for harmonic means, it must also hold for the larger geometric means.

Problem. **"strong polarization"**.

$$\max_{z_1,\dots,z_n\in C} \min_{x\in C} \sum_{j=1}^n |x - z_j|^{-p}.$$

They proved that for $p = 2$ the extremum is taken only for regular n-gons.

Conjecture. **Ambrus, Ball, Erdélyi** For arbitrary $p > 0$, the same holds.

Moreover, denoting by $d(x, y) := d_{\mathbb{T}}(x, y) := \min(|x - y|, 2\pi - |x - y|)$ the distance on the unit circle (torus) $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, if K is any convex "kernel function", then

$$\max_{z_1,\ldots,z_n \in \mathbb{T}} \min_{x \in \mathbb{T}} \sum_{j=1}^{n} K(d_{\mathbb{T}}(z_j, x)).$$

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Saff & al. terms the quantities $|x-y|^{-p}$ as "Riesz kernels" and the corresponding sums as "Riesz potentials", referring to M. Riesz, who investigated such potential kernels on $\mathbb{R}^d$.

Following Fuglede, Ohtsuka, Choquet, Brelot, Deny ...

in general, on say a locally compact topological space X, a kernel function is a lower semi-continuous bivariate function $K : X^2 \to [-\infty, \infty)$.

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The above quantities are the potentials of the sums of n-point Dirac atomic distributions, and the solutions of the n^{th} minimax problem can be identified with the corresponding general Chebyshev constants.

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Even **Chebyshev constants of two sets** were defined in general potential theory: $M_n(A,B) := \inf_{\mu_n \in M_n(A)} \sup_B \int_A K(x,y) d\mu_n(y)$, where $\mu_n \in M_n(A)$ are the averages of n Dirac masses at points located in the set A.

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After taking logarithms, the original **polarization constant problem can be expressed** in terms of general potential theory as looking for

$$M_n(S_{X^\star}, S_X) \qquad \text{with the kernel} \quad K(x,y) := \begin{cases} \log|\langle y; x\rangle| & \text{if} \quad x \in X, y \in X^\star \\ -\infty & \text{otherwise} \end{cases}.$$

On the way solving a problem of Barry in the theory of entire functions, Fenton found the following minimax result.

Theorem (Fenton). *Let J be concave on the interval $I := [0,1]$, and $K \in C^2[-1,1] \setminus \{0\}$, concave and monotonic on $(0,1]$ and $[-1,0)$ with $K" < 0$, satisfying Condition (∞') at 0: $K'_{\pm}(0) = \pm\infty$.*
Then for an extremal configuration in the problems of finding M and m for

$$F(\mathbf{y}, t) = J(t) + \sum_j K(t - y_j)$$

we have equi-oscillation: $m_0 = m_1 = \cdots = m_n$, and the system of extremal nodes $\mathbf{y}$ are unique (for both problems the same), with all y_j different.

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Note that apart from the translated kernels, there is a fixed weight $J(t)$ - an external field - which provided a necessary generality for the application, aimed at.

In 2013 the original conjecture of Ambrus, Ball and Erdélyi was resolved.

Theorem (Hardin, Kendall, Saff). *If K is any concave kernel function on $\mathbb{T}$ and is even (i.e., a function of the distance $d_{\mathbb{T}}$ on the torus) then*

$$\min_{y_j \in \mathbb{T}, j=1,\dots,n} \max_{t \in \mathbb{T}} \sum_{j=1}^{n} K(t - y_j)$$

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What kind of equilibrium do we have here? Consider e.g. the case of most interest, when the kernel is strictly concave with its singularity at zero being infinite ($K(0) = -\infty$). Clearly, then the "potential" $\sum_{j=1}^{n} K(t - y_j)$ is $-\infty$ at each y_j: in the intervals in between, it is strictly concave, and has a unique (local) maximum point z_j and maximum value m_j. The claim is that in **the extremal case** everything **is periodic** by $2\pi/n$ and thus $(m_1, \dots, m_n)$ is a constant vector. Note the strong symmetry here – the singularities ("electrons") – at y_j can be interchanged without any effect.

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In less symmetric cases – say when the kernels, singular at different y_j, are **chosen different**; or when there is some fixed, possibly different **external field** – we **cannot expect periodicity and symmetry** in such an extent.

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Problem. Let $K_0, K_1, \dots, K_n$ be (possibly different) kernel functions on $\mathbb{T}$, and say concave on $(0, 2\pi)$ and singular (in some sense or in another) at 0. Consider the potential function

$$F(\mathbf{y}, t) := K_0(t) + \sum_{j=1}^{n} K_j(t - y_j).$$

Let $I_0, I_1, \dots, I_n$ be the subarcs of $\mathbb{T}$ to which the points $y_j \in \mathbb{T}$ divide $\mathbb{T}$, and $m_j := \sup_{I_j} F(\mathbf{y}, \cdot)$. Characterize the extremal cases when $\inf_{\mathbf{y}\in\mathbb{T}^n} \sup_{t\in\mathbb{T}} F(\mathbf{y}, t)$ is achieved. Study equilibrium properties of $\mathbf{m} = (m_0, m_1, \dots, m_n)$.

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Problem (Bojanov). Assume that a sequence of natural numbers $\nu_1, \nu_2, \ldots, \nu_k$ is given such that $\nu_1 + \cdots + \nu_k = n$. Consider on an interval I the **weighted Chebyshev problem** of placing zeroes of a polynomial of degree n **with the prescribed multiplicities** ν_j such that the ∞-norm would be minimal on I.

This refers to the extremal problem on the interval I

$$\min_{x_1<\cdots<x_k\in I}\ \max_{x\in I}\sum_{j=1}^{k}\nu_j\log|x-x_j|;$$

where we consider $K_j(t):=\nu_j\log|t|$; the potential is $F(\mathbf{x},x):=\sum_{j=1}^{k}\nu_j\log|x-x_j|$.

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Theorem (Bojanov). *Assume that a sequence of natural numbers $\nu_1,\nu_2,\ldots,\nu_k$ is given such that $\nu_1+\cdots+\nu_k=n$. Then there is a unique polynomial on I with minimal sup norm (and with zeroes at $x_1<\ldots<x_k$ with the corresponding multiplicities $\nu_1,\ldots,\nu_k$), and it is exactly the unique polynomial with the local maxima $m_j:=\max_{I_j}F(\mathbf{x},x)$ equi-oscillating (all equal).*

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This was solved by Bojanov on $I:=[-1,1]$ (not on $\mathbb{T}$), by heavy classical type arguments (oscillation of polynomials, zero counting, monotonicity properties in function of the monotonic perturbation of nodes etc.).

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The analogous trigonometrical polynomial case has not been described ever since.

For $n \in \mathbb{N}$ and $j = 0, \ldots, n$ let K_j be a strictly concave kernel function on $(0, 2\pi)$ that has an infinite cusp at $0 \in \mathbb{T}$, i.e., it is such that $\lim_{\substack{t\to 0\\ t\in\mathbb{T}}} K_j(t) = -\infty$ meaning that

$$\lim_{t\searrow 0} K_j(t) = -\infty = \lim_{t\nearrow 2\pi} K_j(t). \qquad (\infty)$$

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Denote by $D_- f$ and $D_+ f$ the left and right derivatives of a function f defined on an interval, respectively. A **concave** function f, defined on an open interval possesses at each points left and right derivatives, and $D_- f$, $D_+ f$ are decreasing functions.

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$$\left.\begin{aligned} \lim_{t\searrow 0} K_j(t) = \lim_{t\nearrow 2\pi} K_j(t) =: K_j(0), \\ \text{and } \lim_{t\nearrow 2\pi} D_+ K_j(t) = \lim_{t\nearrow 2\pi} D_- K_j(t) = -\infty, \\ \text{and } \lim_{t\searrow 0} D_- K_j(t) = \lim_{t\searrow 0} D_+ K_j(t) = \infty \end{aligned}\right\} \qquad (\infty')$$

(equivalently written in the form $D_\pm K_j(0) = \pm\infty$).

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To distinguish from the case of an actual singularity, this, for concave kernels less restrictive condition than (∞), will be spelled out as the concave kernel functions K_j having a (finite) **cusp** (at $0 \in \mathbb{T}$).

For a permutation σ of $\{1, \dots, n\}$ we introduce $\sigma(0) := 0$ and $\sigma(n+1) := n+1$, and we define the **simplex**

$$S_\sigma := \big\{\mathbf{y} \in \mathbb{T}^n : 0 = y_{\sigma(0)} < y_{\sigma(1)} < \cdots < y_{\sigma(n)} < y_{\sigma(n+1)} = 2\pi\big\}.$$

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Then the S_σ are open, disjoint subsets of $\mathbb{T}^n$ with

$$\bigcup_\sigma \overline{S}_\sigma =: \mathbb{T}^n \quad \text{and for} \quad X := \bigcup_\sigma S_\sigma \quad \text{we have} \quad \operatorname{int}(\mathbb{T}^n \setminus X) = \emptyset;$$

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Given σ and $\mathbf{y} \in \overline{S}_\sigma$, for $k = 0,\ldots,n$ we define **the k^{th} arc $I_{\sigma(k)}$** as

$$I_{\sigma(k)}(\mathbf{y}) := [y_{\sigma(k)}, y_{\sigma(k+1)}].$$

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For an arbitrary $\mathbf{y} \in \mathbb{T}^n$ the corresponding **numbered sequence** $I_j(\mathbf{y})$ of arcs are defined unambigously, **as soon as we specify** σ with $\mathbf{y} \in \overline{S}_\sigma$.

For $\sigma \neq \pi$ and for $\mathbf{y} \in \overline{S}_\sigma \cap \overline{S}_\pi$ on the (common) boundary, the **system of arcs** is still well defined, but their **numbering** depends on the permutations π and σ.

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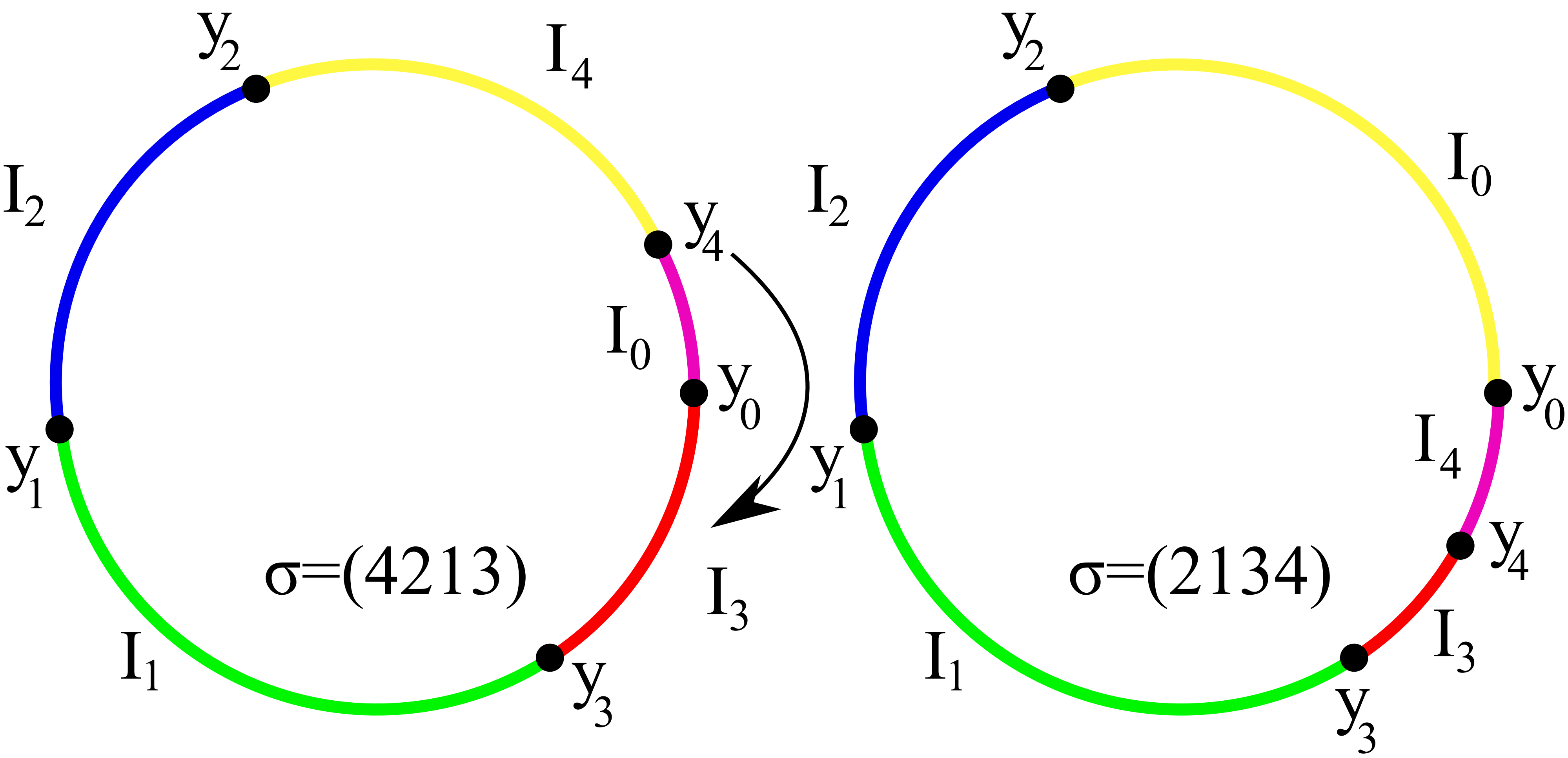

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Of interest are then the following two minimax type expressions:

$$M := \inf_{\mathbf{y} \in \mathbb{T}^n} \overline{m}(\mathbf{y}) = \inf_{\mathbf{y} \in \mathbb{T}^n} \max_{j=0,\dots,n} m_j(\mathbf{y}) = \inf_{\mathbf{y} \in \mathbb{T}^n} \sup_{t \in \mathbb{T}} F(\mathbf{y}, t), \tag{1}$$

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Or more specifically for any given simplex $S = S_\sigma$ we may consider the problems:

$$M(S) := \inf_{\mathbf{y} \in S} \overline{m}(\mathbf{y}) = \inf_{\mathbf{y} \in S} \max_{j=0,\ldots,n} m_j(\mathbf{y}) = \inf_{\mathbf{y} \in S} \sup_{t \in \mathbb{T}} F(\mathbf{y}, t), \tag{3}$$

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For notational convenience for any given set $A \subseteq \mathbb{T}^n$ we also define

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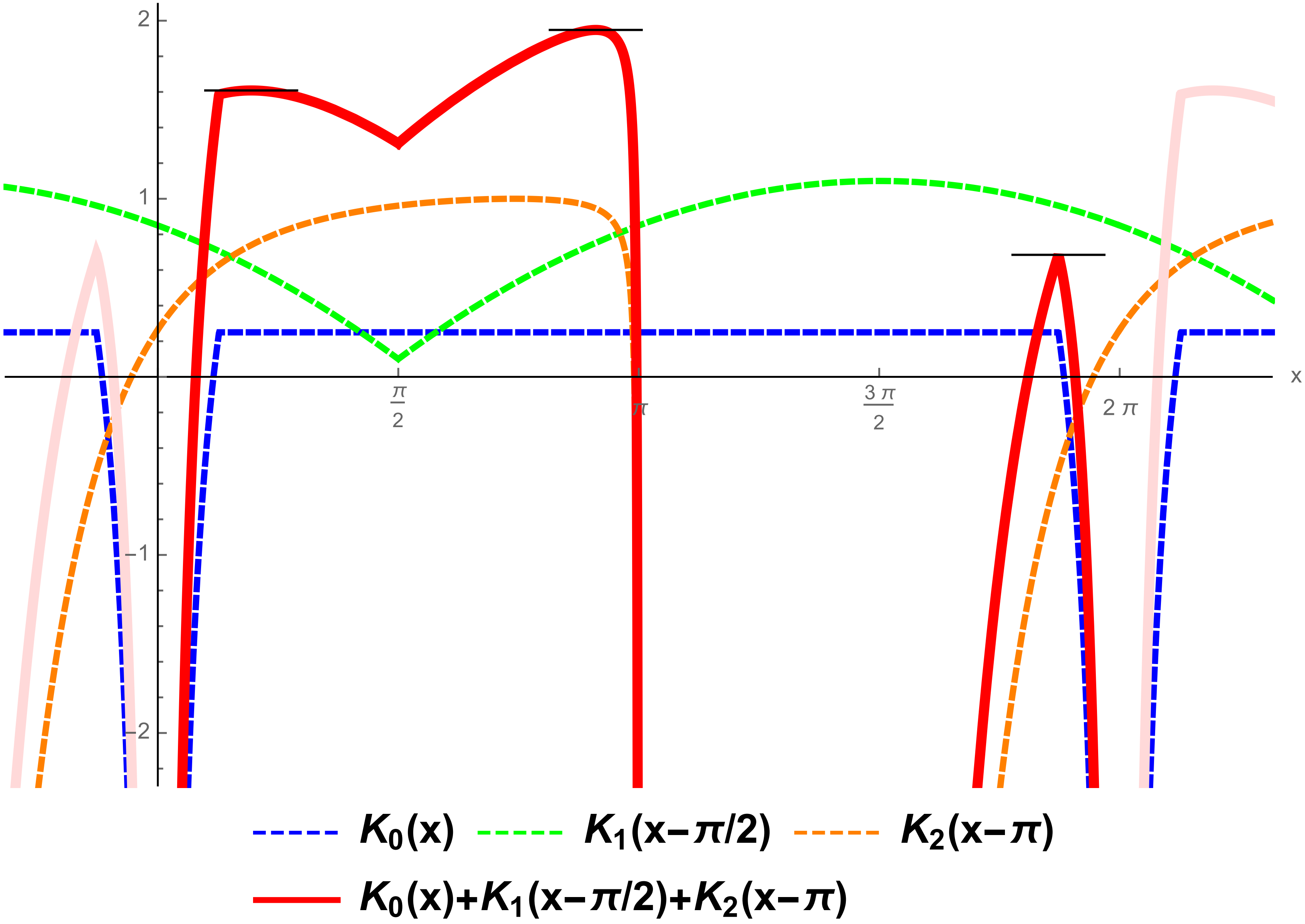
We are interested in whether the infimum or supremum are always attained, and if so, what can be said about the extremal configurations.

Example. *If the kernels are only concave and not strictly concave, then the minimax problem* (3) *may have many solutions, even on the boundary* ∂S *of* $S = S_\sigma$*.* *Let* $n \in \mathbb{N}$, $\delta < \frac{\pi}{n+1}$ *and let* K *be an even* ($K(t) = K(2\pi - t)$) *kernel which is constant* c_0 *on the interval* $[\delta, 2\pi - \delta]$ *and* $K(t) = -1/t + 1/\delta + c_0$ *if* $0 < t \leq \delta$*. Put* $K_0 = K_1 = \cdots = K_n = K$*.*

Then for any node system $\mathbf{y}$ *we have* $\max_{t\in\mathbb{T}} F(\mathbf{y}, t) = (n+1)c_0$*, because the* 2δ *long intervals around the nodes cannot cover* $[0, 2\pi]$*.*


2
1
−1
−2
x
π/2
π
3π/2
2π
$K_0(x)$
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Proposition. *For every* $\delta > 0$ *there is* $L = L(K_0, \ldots, K_n, \delta) \geq 0$ *such that for every* $\mathbf{y} \in \mathbb{T}^n$ *and for every* $j \in \{0, \ldots, n\}$ *with* $|I_j(\mathbf{y})| > \delta$ *one has* $m_j(\mathbf{y}) \geq -L$*.*

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For $\delta := \frac{2\pi}{n+2}$ take $L \geq 0$ as in Proposition . Then for every $\mathbf{y} \in \mathbb{T}^n$ there is $j \in \{0, \dots, n\}$ with $|I_j(\mathbf{y})| > \delta$, so that for this j we have $m_j(\mathbf{y}) \geq -L$. This implies $M(S) \geq -L > -\infty$.

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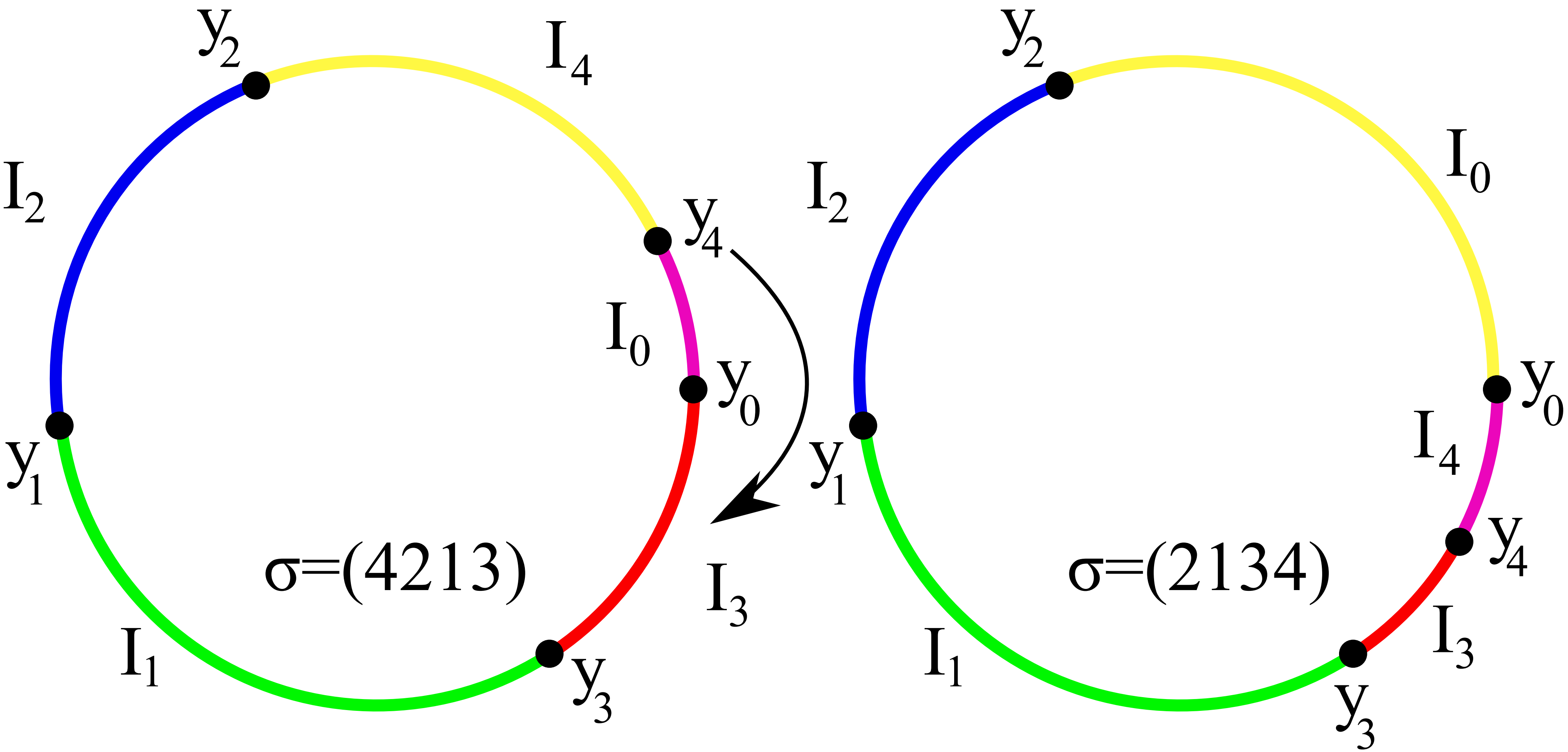

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So in a (uniform-) δ-neighborhood $U := U(\mathbf{y}_0, \delta) := \{\mathbf{x} \in \mathbb{T}^n \ : \ \|\mathbf{x} - \mathbf{y}_0\|_\infty < \delta\}$ of $\mathbf{y}_0 \in \mathbb{T}^n$, **no node can reach** c. We **cut the torus at** c and represent $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ by the interval $[c, c+2\pi) \sim [0, 2\pi)$, and we use the ordering of this interval.

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$$\ell_i(\mathbf{y}) := \min\{t \in [c, c+2\pi) \; : \; \#\{k : y_k \le t\} \ge i\}$$
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Obviously, the system of arcs $\{I_j \ : \ j = 0, \ldots, n\}$ is the same as $\{\widehat{I_i} \ : \ i = 0, \ldots, n\}$.

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Now the condition (∞) means that the kernels $K_j : (0, 2\pi) \to \mathbb{R}$ can be continuously extended to $[0, 2\pi]$ having then values in $\bar{\mathbb{R}}$.

Proposition. *Suppose the kernels are continuous (to $\bar{\mathbb{R}}$) and bounded from above (do not take the value ∞). Then*

$$F : \mathbb{T}^n \times \mathbb{T} \to [-\infty, \infty)$$

is uniformly continuous (to $\bar{\mathbb{R}}$, in the above sense).

Proposition. *Suppose the kernels are continuous bounded from above. Let* $\mathbf{y}_0 \in \mathbb{T}^n$ *be a node system and let* c *be a suitable point of cut of the torus. Then for* $i = 0, \ldots, n$ *the functions*

$$\mathbf{y} \mapsto \widehat{m}_i(\mathbf{y}) := \sup_{t \in \widehat{I}_i(\mathbf{y})} F(\mathbf{y}, t) \in [-\infty, \infty]$$

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Hence $f_i(\mathbf{y}) := \max_{t \in \widehat{I}_i(\mathbf{y})} \arctan \circ F(\mathbf{y}, t)$ (and thus also $\widehat{m_i} = \tan \circ f_i$) is continuous.

The continuity of $\widehat{m_i}$ for fixed i involves the cut of the torus at c. However, if **for the system** $\{m_0, \ldots, m_n\} = \{\widehat{m_0}, \ldots, \widehat{m_n}\}$, the dependence on the cut can be cured.

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For $\mathbf{x} \in \mathbb{T}^{n+1}$ define

$$T_i(\mathbf{x}) := \min\{t \in [c, c+2\pi) \ : \ \exists k_0, \ldots, k_i \text{ s.t. } x_{k_0}, \ldots, x_{k_i} \leq t\} \quad (i = 0, \ldots, n)$$

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Corollary. *Suppose the kernels are continuous and do not take the value ∞. The mapping*

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Proof. We have $T(m_0(\mathbf{y}), \ldots, m_n(\mathbf{y})) = T(\widehat{m}_0(\mathbf{y}), \ldots, \widehat{m}_n(\mathbf{y}))$ for any $\mathbf{y} \in \mathbb{T}$, while $\mathbf{y} \mapsto (\widehat{m}_0(\mathbf{y}), \ldots, \widehat{m}_n(\mathbf{y}))$ is continuous at any given point $\mathbf{y}_0 \in \mathbb{T}^n$ and for any given cut. But the left-hand term here does not depend on the cut. We are done.

Corollary. *Suppose the kernels are continuous and do not take the value ∞. The functions*

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Proof. Let $\mathbf{y}_0 \in \overline{S}$, then there is a cut at some c and there is some i, such that we have $m_j(\mathbf{y}) = \widehat{m}_i(\mathbf{y})$ for all $\mathbf{y}$ in $U \cap \overline{S}$ for a small neighborhood U of $\mathbf{y}_0$. So the continuity follows from the Proposition .

Remark. Suppose that the kernel functions are concave and at least one of them is strictly concave. For fixed $\mathbf{y}$ also $F(\mathbf{y}, \cdot)$ is **strictly concave** on the interior of each arc $I_j(\mathbf{y})$ and continuous on $I_j(\mathbf{y})$, so there is a **unique** $z_j(\mathbf{y}) \in I_j(\mathbf{y})$ with

$$m_j(\mathbf{y}) = F(\mathbf{y}, z_j(\mathbf{y}))$$

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If condition (∞) holds, then it is evident that $z_j(\mathbf{y})$ belongs to the interior of $I_j(\mathbf{y})$ (if this latter is non-empty). However, we obtain the same even under the weaker assumption (∞').

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Proposition. *Suppose that $K_0, \ldots, K_n$ are concave with at least one of them strictly, and continuous (in the extended sense). Then for each $\mathbf{y} \in \mathbb{T}$ and $j = 0, \ldots, n$ there is a unique maximum point $z_j(\mathbf{y})$ of $m_j(\mathbf{y})$ in $I_j(\mathbf{y})$.*

Moreover, if condition (∞') holds for each $j = 0, \ldots, n$, then $z_j(\mathbf{y})$ belongs to the interior of $I_j(\mathbf{y})$ whenever it is non-degenerate.

Proof. Uniqueness of a maximum point $z_j(\mathbf{y})$ has been already discussed.

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To prove that e.g. $z_j \neq y_j$, it suffices to refer to the fact that $D_+F(t,\mathbf{y}) = \sum_{i=0}^{n} D_+K_j(t-y_i) \to -\infty$ when $t \to y_j+0$, whence $F(\cdot,\mathbf{y})$ is increasing in $y_j, y_j+\delta]$.

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Next we need that z_j **are well-defined**, so we need $F(\mathbf{y},\cdot)$ to be strictly concave – it suffices if at least **one of the kernels is strictly concave**.

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Next we need that z_j **are well-defined**, so we need $F(\mathbf{y},\cdot)$ to be strictly concave – it suffices if at least **one of the kernels is strictly concave**.

Lemma. *Suppose that the kernels are concave and continuous, with at least one of them strictly concave, so that $z_j(\mathbf{y}) \in I_j$ is unique for every $j = 0,\ldots,n$. For each $j = 0,\ldots,n$ and for each simplex $S = S_\sigma$ the mapping*

$$z_j : \overline{S} \to \mathbb{T}, \quad \mathbf{y} \mapsto z_j(\mathbf{y})$$

is continuous; furthermore, this continuity remains in effect as long as I_j is changing continuously as a function of the node system $\mathbf{y} \in \mathbb{T}^n$.

Moreover, for a given $\mathbf{y}_0 \in \mathbb{T}^n$ consider a cut at c. Then the mapping

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Corollary. *The system of maximum points, as well as the system of maximum values on the arcs—as determined by the node systems* $\mathbf{y} \in \mathbb{T}^n$*—change continuously all over* $\mathbb{T}^n$*. That is, putting* $\mathbf{z} := (z_0, \ldots, z_n)$ *and* $\mathbf{m} := (m_0, \ldots, m_n)$*, we have* $T(\mathbf{z}(\mathbf{y})) \in C(\mathbb{T}^{n+1} \to \mathbb{T}^{n+1})$*,* $T(\mathbf{m}(\mathbf{y})) \in C(\mathbb{R}^{n+1} \to \mathbb{R}^{n+1})$*.*

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Proposition. *For a simplex* S *we always have* $M(S) = M(\overline{S})$ *and* $m(S) = m(\overline{S})$*. Furthermore, both minimax problems* (3) *and* (4) *have finite extremal values, and both have an extremal node system, i.e., there are* $\mathbf{w}^*, \mathbf{w}_* \in \overline{S}$ *such that*

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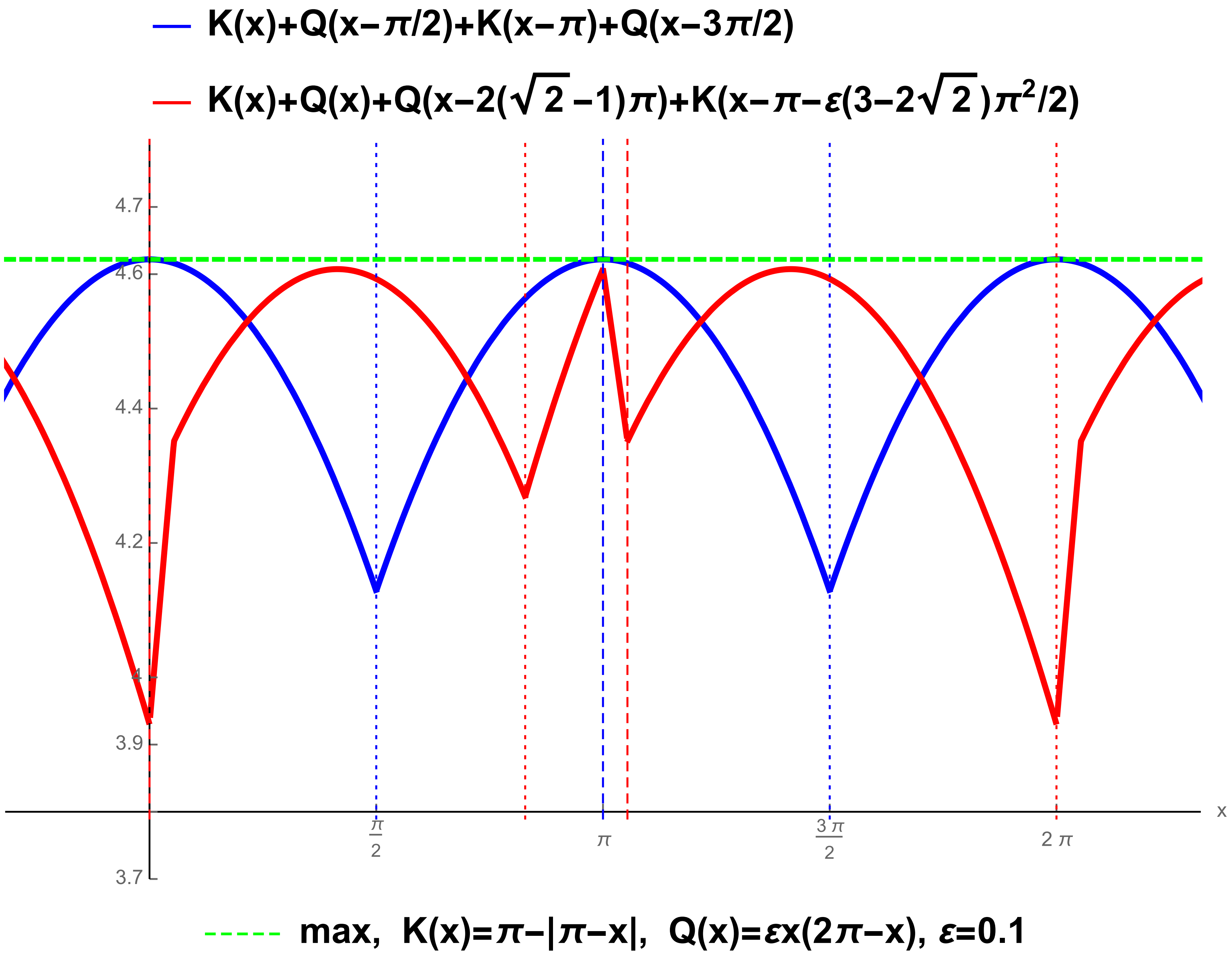
Corollary. *Both minimax problems* (1) *and* (2) *have an extremal node system.*

Remark. However surprising, but it can well happen, that the extremal problems on different simplexes have different extremal values. We have an example. In this case, some simplex admits extremal values on the boundary of the simplex.


K(x)+Q(x−π/2)+K(x−π)+Q(x−3π/2)
K(x)+Q(x)+Q(x−2(√2−1)π)+K(x−π−ε(3−2√2)π²/2)
4.7
4.6
4.4
4.2
4
3.9
3.7
x
π/2
π
3π/2
2π
max, K(x)=π−|π−x|, Q(x)=εx(2π−x), ε=0.1

In what follows we shall consider a sequence $K_j^{(k)}$ of kernel functions converging to K_j as $k \to \infty$ for $j = 0, \ldots, n$ (in some sense or another).

The corresponding values of local maxima and related quantities will be denoted as $m_j^{(k)}(\mathbf{x})$, $\underline{m}^{(k)}(\mathbf{x})$, $\overline{m}^{(k)}(\mathbf{x})$, $m^{(k)}(S)$, $M^{(k)}(S)$.

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Let K be a compact space and let $f_n, f \in \mathrm{C}(K; \bar{\mathbb{R}})$ (the space of continuous functions with values in $\bar{\mathbb{R}}$).

We say that $f_n \to f$ **uniformly** (in the extended sense) if $\arctan f_n \to \arctan f$ uniformly in the ordinary sense (as real valued functions).

We say that $f_n \to f$ **strongly uniformly** if for all $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$f(x) - \varepsilon \leq f_n(x) \leq f(x) + \varepsilon \quad \text{for every } x \in \mathbb{T} \text{ and } n \geq n_0.$$

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Lemma. *Let $f_n, g_n, f, g \in \mathrm{C}(K;\bar{\mathbb{R}})$ for $n \in \mathbb{N}$.*

1. *If $f_n, g_n \le C < \infty$ and $f_n \to f$ and $g_n \to g$ uniformly (in the extended sense), then $f_n + g_n \to f + g$ uniformly (in the extended sense).*
2. *If $f_n \searrow f$ pointwise, i.e., if $f_n(x) \to f(x)$ decreasingly, then $f_n \to f$ uniformly.*
3. *If $f_n \to f$ uniformly, then $\sup f_n \to \sup f$ in $[-\infty, \infty]$.*

Proposition. *Suppose the sequence of kernel functions* $K_j^{(k)} \to K_j$ *uniformly for* $k \to \infty$ *and* $j = 0, 1, \ldots, n$*. Then for each simplex* $S := S_\sigma$ *we have that* $m_j^{(k)} \to m_j$ *uniformly on* $\bar{S}$ *(*$j = 0, 1, \ldots, n$*). As a consequence,* $m^{(k)}(S) \to m(S)$ *and* $M^{(k)}(S) \to M(S)$ *as* $k \to \infty$*.*

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Now let us define the compact set – the **diagonal** set of F – as

$$D := \left\{ (\mathbf{x}, t) \ : \ \exists i \in \{0, 1, \ldots, n\}, \text{ such that } t = x_i \right\} = \bigcup_{i=0}^{n} \left\{ (\mathbf{x}, t) \ : \ t = x_i \right\} \subseteq \mathbb{T}^{n+1}.$$

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Lemma. *Assume that the sequence of kernel functions $K_j^{(k)}$ converges to the kernel function K_j* **locally uniformly** *on $(0, 2\pi)$.*
Then $F^{(k)}(\mathbf{x}, t) \to F(\mathbf{x}, t)$ **locally uniformly** *on $\mathbb{T}^{n+1} \setminus D$, i.e., for every compact subset $H \subseteq \mathbb{T}^{n+1} \setminus D$ one has $F^{(k)}(\mathbf{x}, t) \to F(\mathbf{x}, t)$ uniformly on H as $k \to \infty$.*

Proposition. *Suppose the sequence of kernel functions* $K_j^{(k)} \to K_j$ *uniformly for* $k \to \infty$ *and* $j = 0, 1, \ldots, n$. *Then for each simplex* $S := S_\sigma$ *we have that* $m_j^{(k)} \to m_j$ *uniformly on* $\bar{S}$ *(*$j = 0, 1, \ldots, n$*). As a consequence,* $m^{(k)}(S) \to m(S)$ *and* $M^{(k)}(S) \to M(S)$ *as* $k \to \infty$.

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Lemma. *Assume that the sequence of kernel functions* $K_j^{(k)}$ *converges to the kernel function* K_j **locally uniformly** *on* $(0, 2\pi)$.
Then $F^{(k)}(\mathbf{x}, t) \to F(\mathbf{x}, t)$ **locally uniformly** *on* $\mathbb{T}^{n+1} \setminus D$, *i.e., for every compact subset* $H \subseteq \mathbb{T}^{n+1} \setminus D$ *one has* $F^{(k)}(\mathbf{x}, t) \to F(\mathbf{x}, t)$ *uniformly on* H *as* $k \to \infty$.

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Then $F^{(k)}(\mathbf{x}, t) \to F(\mathbf{x}, t)$ **locally uniformly** *on* $\mathbb{T}^{n+1} \setminus D$*, i.e., for every compact subset* $H \subseteq \mathbb{T}^{n+1} \setminus D$ *one has* $F^{(k)}(\mathbf{x}, t) \to F(\mathbf{x}, t)$ *uniformly on* H *as* $k \to \infty$*.*

Note that F can attain $-\infty$, and that convergence of $K_j^{(k)}(0)$ is not postulated.

Corollary. *Suppose that* $z \neq x_j$*,* $j = 0, \ldots, n$*, whenever* $F(\mathbf{x}, z) = \overline{m}(\mathbf{x})$*. If the sequence of kernel functions* $K_j^{(k)} \to K_j$ *locally uniformly on* $(0, 2\pi)$*, then* $\overline{m}^{(k)}(\mathbf{x}) \to \overline{m}(\mathbf{x})$ *uniformly on* $\mathbb{T}^n$*.*

Proposition. *Suppose that all the K_j satisfy* (∞). *Let $S = S_\sigma$ be a simplex. Then*

$$\lim_{\mathbf{y}\to\partial S} \max_{j=0,\dots,n-1} \left| m_{\sigma(j)}(\mathbf{y}) - m_{\sigma(j+1)}(\mathbf{y}) \right| = \infty. \tag{7}$$

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The properties introduced below have nothing to do with the conditions we pose on the kernel functions $K_0, \ldots, K_n$ (concavity and some type of singularity at 0 and 2π), so we can formulate them in whole generality. (Note that m_j (in contrast to z_j) is well-defined even if the kernels are not strictly concave).

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Definition. Let $S = S_\sigma$ be a simplex.

(a) **Jacobi Property:**

The functions $m_0, \ldots, m_n$ are C^1 on $S = S_\sigma$ and

$$\det\Big(\partial_i m_{\sigma(j)}\Big)_{i=1, j=0, j\neq k}^{n,n} \neq 0 \quad \text{for each } k \in \{0, \ldots, n\}.$$

(b) **Difference Jacobi Property:**

The functions $m_0, \ldots, m_n$ are C^1 on S and

$$\det\Big(\partial_i (m_{\sigma(j)} - m_{\sigma(j+1)})\Big)_{i=1, j=0}^{n,n-1} \neq 0.$$

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(a) **Equioscillation Property:**
There exists an **equioscillation point** $\mathbf{y} \in S$, i.e.,

$$\overline{m}(\mathbf{y}) = \underline{m}(\mathbf{y}) = m_0(\mathbf{y}) = m_1(\mathbf{y}) = \cdots = m_n(\mathbf{y}).$$

(b) **(Lower) Weak Equioscillation Property:**
There exists a **weak equioscillation point** $\mathbf{y} \in \overline{S}$, i.e., for all j

$$m_j(\mathbf{y}) \begin{cases} = \overline{m}(\mathbf{y}), & \text{if } I_j \text{ is non-degenerate,} \\ < \overline{m}(\mathbf{y}), & \text{if } I_j \text{ is degenerate.} \end{cases}$$

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Remark. For given $S = S_\sigma$ the Equioscillation Property implies the inequality $M(S) \leq m(S)$. Indeed, if $\mathbf{y} \in S$ is an equioscillation point, then $\overline{m}(\mathbf{y}) = \max_{j=0,\ldots,n} m_j(\mathbf{y}) = \min_{j=0,\ldots,n} m_j(\mathbf{y}) = \underline{m}(\mathbf{y})$, whence $M(S) \leq \overline{m}(\mathbf{y}) = \underline{m}(\mathbf{y}) \leq m(S)$.

Proposition. *Given a simplex $S = S_\sigma$ the followings are equivalent:*

(i) $M(S) \geq m(S)$.

(ii) *For every $\mathbf{x} \in S$ one has $\underline{m}(\mathbf{x}) = \min_{j=0,\ldots,n} m_j(\mathbf{x}) \leq M(S)$.*

(iii) *For every $\mathbf{y} \in S$ one has $\overline{m}(\mathbf{y}) = \max_{j=0,\ldots,n} m_j(\mathbf{y}) \geq m(S)$.*

(iv) *There exists a value $\mu \in \mathbb{R}$ such that for each $\mathbf{y} \in S$*

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Proof. Recalling the inequalities

$$\overline{m}(\mathbf{y}) = \max_{j=0,\dots,n} m_j(\mathbf{y}) \geq M(S) = \inf_S \overline{m}, \quad \sup_S \underline{m} = m(S) \geq \underline{m}(\mathbf{x}) = \min_{j=0,\dots,n} m_j(\mathbf{x})$$

being true for each $\mathbf{x}, \mathbf{y} \in S$, the equivalence of (i), (ii) and (iii) is obvious. Suppose (i) and take $\mu \in [m(S), M(S)]$. Then (iv) is evident. From (iv) (i) follows trivially.

Definition. Let $S = S_\sigma$ be a simplex. We say that the **Sandwich Property** is satisfied if any of the equivalent assertions in the Proposition holds true, i.e., if

$$\max_{j=0,\dots,n} m_j(\mathbf{y}) = \overline{m}(\mathbf{y}) \geq \underline{m}(\mathbf{x}) = \min_{j=0,\dots,n} m_j(\mathbf{x}). \quad (\mathbf{x}, \mathbf{y} \in S)$$

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Remark. For given $S = S_\sigma$ the Equioscillation Property and the Sandwich Property together imply that $M(S) = m(S)$.

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Remark. For given $S = S_\sigma$ the Equioscillation Property and the Sandwich Property together imply that $M(S) = m(S)$.

Definition. We say that x **majorizes** (or **strictly majorizes**) y—and y **minorizes** (or **strictly minorizes**) x—if $m_j(\mathbf{x}) \geq m_j(\mathbf{y})$ (or if $m_j(\mathbf{x}) > m_j(\mathbf{y})$) for all $j = 0, \ldots, n$.

Let $S = S_\sigma$ be a simplex. We define the following properties on S_σ.

Let $S = S_\sigma$ be a simplex. We define the following properties on S_σ.

(a) **Local (Strict) Comparison Property at $\mathbf{z}$:** There exists $\delta > 0$ such that if $\mathbf{x}, \mathbf{y} \in B(\mathbf{z}, \delta)$ and $\mathbf{x}$ (strictly) majorizes $\mathbf{y}$, then $\mathbf{x} = \mathbf{y}$. In other words, there are no two different $\mathbf{x} \neq \mathbf{y} \in B(\mathbf{z}, \delta)$ with $\mathbf{x}$ (strictly) majorizing $\mathbf{y}$.

(b) **Local (Strict) Non-Majorization Property at $\mathbf{y}$:**
There exists $\delta > 0$ such that there is no $\mathbf{x} \in (S \cap B(\mathbf{y}, \delta)) \setminus \{\mathbf{y}\}$ which (strictly) majorizes $\mathbf{y}$.

(c) **Local (Strict) Non-Minorization Property at $\mathbf{y}$:**
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Further, we will pick the following special cases as important.

(A) **(Strict) Comparison Property:**
If $\mathbf{x}, \mathbf{y} \in S$ and $\mathbf{x}$ (strictly) majorizes $\mathbf{y}$, then $\mathbf{x} = \mathbf{y}$. In other words, there exists no two different $\mathbf{x} \neq \mathbf{y} \in S$ with $\mathbf{x}$ (strictly) majorizing $\mathbf{y}$.

(B) **Local (Strict) Comparison Property on S:** At each point $\mathbf{z} \in \overline{S}$, the Local (Strict) Comparison Property holds.

(C) **Local (Strict) Non-Majorization Property on S:**
At each point $\mathbf{z} \in \overline{S}$, the Local (Strict) Non-Majorization Property holds.

(D) **Local (Strict) Non-Minorization Property on S:**
At each point $\mathbf{z} \in \overline{S}$, the Local (Strict) Non-Minorization Property holds.

(E) **Singular (Strict) Comparison Property on S:** At each **equioscillation point** $\mathbf{y} \in S$ the Local (Strict) Comparison Property holds.

(F) **Singular (Strict) Non-Majorization Property:** At each **equioscillation point** $\mathbf{y} \in S$ the Local (Strict) Non-Majorization Property holds.

(G) **Singular (Strict) Non-Minorization Property:** At each **equioscillation point** $\mathbf{y} \in S$ the Local (Strict) Non-Minorization Property holds.

Remark. The comparison properties are symmetric in $\mathbf{x}$ and $\mathbf{y}$, while the non-majorization and non-minorization properties are not.

One has the following relations between the above defined properties: (a)⇒(b) and (c), (A)⇒(B)⇒(E), (B)⇒(C) and (D), (E)⇒(F) and (G), (C)⇒(F), (D)⇒(G).

It can be proved that for strictly concave kernels **all comparison, non-majorization and non-minorization properties (A), (B), (C), (D) (with their strict version as well) are equivalent** to each other.

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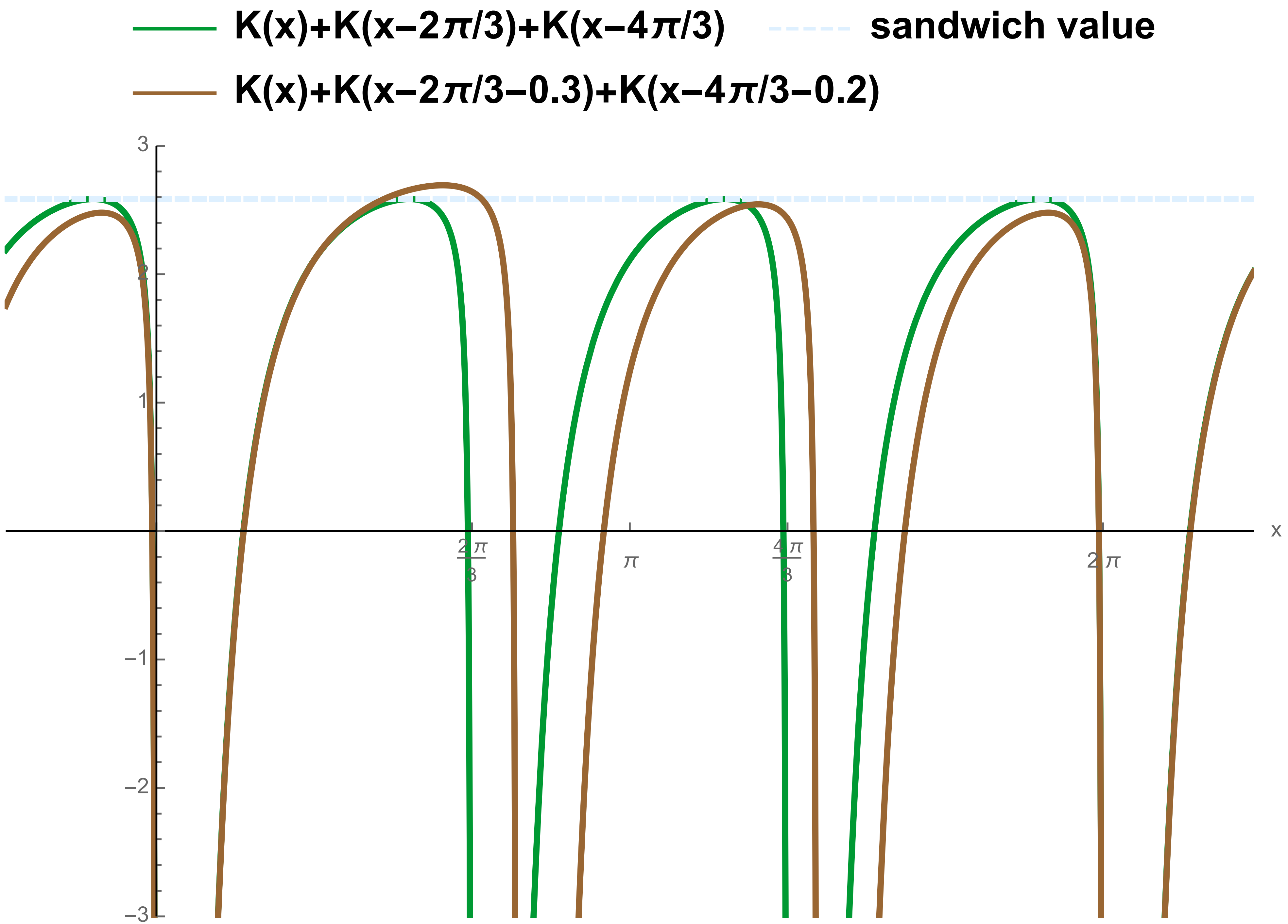
It can be proved that for strictly concave kernels **all comparison, non-majorization and non-minorization properties (A), (B), (C), (D) (with their strict version as well) are equivalent** to each other.

Remark. Shi proved that (under condition (7)) the Jacobi Property implies the Comparison Property and the Sandwich Property, and that the Difference Jacobi Property implies the Equioscillation Property.

The Example below shows that the Comparison Property (even the Local Strict Non-Majorization Property) – whence also the Jacobi Property – fails in general, even though the Difference Jacobi Condition is fulfilled.

We will show that in our setting the Difference Jacobi Condition is always fulfilled (as long as the kernels are at least C^2) and so we have the Equioscillation Property.



K(x)+K(x−2π/3)+K(x−4π/3)
sandwich value
K(x)+K(x−2π/3−0.3)+K(x−4π/3−0.2)
3
2
1
−1
−2
−3
x
2π/3
π
4π/3
2π

Example. *Let* $n = 1$ *and* $K_0 : (0, 2\pi) \to \mathbb{R}$ *be a strictly concave* C^∞ *kernel function satisfying* (∞) *and such that the maximum of* K_0 *is* 0, K_0 *is increasing* $(0, \alpha)$ *with* $\alpha \in (0, \pi)$ *and is decreasing in* $(\alpha, 2\pi)$, *and let* $K_1(t) := K_0(2\pi - t)$.

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\begin{aligned}
m_0(\mathbf{y}) &= F(\mathbf{y}, z_0(\mathbf{y})) = K_0(\tfrac{y}{2}) + K_0(2\pi + y - \tfrac{y}{2}) = 2K_0(\tfrac{y}{2}), \\
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Whence we conclude that

$$m_0(\mathbf{y} + h) < m_0(\mathbf{y}) \quad \text{and} \quad m_1(\mathbf{y} + h) < m_1(\mathbf{y}),$$

whenever $y \in (2\alpha, 1)$ and $h > 0$ with $y + h \in (2\alpha, 1)$.

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This shows that the Non-Majorization Property does not hold in general.

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This shows that the Non-Majorization Property does not hold in general.

Since $m_0'(2\alpha) = 0$, the Jacobi Property fails for this example (which anyway follows from the above Remark).

Notice also that

$$m_0'(\mathbf{y}) - m_1'(\mathbf{y}) = K_0'(\tfrac{y}{2}) - K_0'(\tfrac{2\pi+y}{2}) > 0,$$

since K_0' is strictly decreasing, meaning that the Difference Jacobi Condition holds (this holds in general, see Proposition **??**).

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Finally, we remark that we have here the Singular Non-Majorization Property. Indeed, $\mathbf{y}$ is an equioscillation point if and only if

$$2K_0(\tfrac{y}{2}) = m_0(\mathbf{y}) = m_1(\mathbf{y}) = 2K_0(\tfrac{2\pi+y}{2}),$$

i.e., at the corresponding points in the graph of K_0 there is a chord of length π. This implies that $y/2$ falls in the interval where K_0 is strictly increasing, whereas $\pi + y/2$ belongs to the interval where K_0 is strictly decreasing. Hence if we move $\mathbf{y} = y$ slightly, m_0 and m_1 will change in different directions.

Notice also that

$$m_0'(\mathbf{y}) - m_1'(\mathbf{y}) = K_0'(\tfrac{y}{2}) - K_0'(\tfrac{2\pi+y}{2}) > 0,$$

since K_0' is strictly decreasing, meaning that the Difference Jacobi Condition holds (this holds in general, see Proposition **??**).

Finally, we remark that we have here the Singular Non-Majorization Property. Indeed, $\mathbf{y}$ is an equioscillation point if and only if

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This example shows that Shi's results are not applicable in this general setting, even if we supposed the kernels to be C^∞.

Lemma. *Suppose the kernels $K_0, \ldots, K_n$ are strictly concave and either all satisfy (∞') or all belong to $\mathrm{C}^1(0, 2\pi)$. Let $j \in \{0, \ldots, n\}$ and $\mathbf{w} \in \mathbb{T}^n$ be such that $m_j(\mathbf{w}) = \overline{m}(\mathbf{w})$. Then $z_j(\mathbf{w})$ belongs to the interior of $I_j(\mathbf{w})$.*

Lemma. *Suppose the kernels $K_0, \ldots, K_n$ are strictly concave and either all satisfy (∞') or all belong to $\mathrm{C}^1(0, 2\pi)$. Let $j \in \{0, \ldots, n\}$ and $\mathbf{w} \in \mathbb{T}^n$ be such that $m_j(\mathbf{w}) = \overline{m}(\mathbf{w})$. Then $z_j(\mathbf{w})$ belongs to the interior of $I_j(\mathbf{w})$.*

Remark. Suppose f_j are (strictly) concave functions for $j = 0, \ldots, n$ and let $f = \sum_{j=0}^{n} f_j$. Let μ_j be the slope of a tangent line of f_j at some point t, then $\mu := \sum_{j=0}^{n} \mu_j$ is the slope of a tangent line of f at the same point t. Conversely, if μ is given as a slope of a tangent line at some point t, then it is not hard to find μ_j, $j = 0, \ldots, n$ being the slope of the corresponding tangent line of f_j at t, with $\mu = \sum_{j=0}^{n} \mu_j$.

Lemma (Perturbation lemma). *Suppose that $K_0, \ldots, K_n$ are strictly concave. Let $\mathbf{y} \in \mathbb{T}^n$ be a node system, and for $k \in \mathbb{N}$, $1 \leq k \leq n$ let $t_1, \ldots, t_k \in (0, 2\pi)$ be all different from the nodes in $\mathbf{y}$. Let*

$$\delta := \frac{1}{2} \min\left\{ |t_i - y_j| : i = 1, \ldots, k,\ j = 0, \ldots, n \right\}$$

For $i = 1, \ldots, k$ let $\mu^{(i)}$ be the slope of a tangent line to the graph of $F(\mathbf{y}, \cdot)$ at the point t_i. Finally, let $\mathbf{x}_1, \ldots, \mathbf{x}_{n-k} \in \mathbb{R}^n$ be fixed arbitrarily.

Lemma (Perturbation lemma - cont'd).

(a) *Then there is* $\mathbf{a} \in [-1,1]^n \setminus \{\mathbf{0}\}$ *such that* $\mathbf{x}_\ell^\top \mathbf{a} = 0$ *for* $\ell = 1, \dots, n-k$ *and for all* $0 < h < \delta$ *we have*

$$F(\mathbf{y} + h\mathbf{a}, s_i) < F(\mathbf{y}, t_i) + \mu^{(i)}(s_i - t_i)$$

for all s_i *with* $|s_i - t_i| < \delta$, $i = 1, \dots, k$.

(b) *If* $F(\mathbf{y}, \cdot)$ *has local maximum in* t_i *for some* $i \in \{1, \dots, k\}$, *i.e., when* $t_i = z_j(\mathbf{y}) \in \operatorname{int} I_j(\mathbf{y})$ *for some* $j \in \{0, \dots, n\}$, *then*

$$F(\mathbf{y} + h\mathbf{a}, s_i) < F(\mathbf{y}, z_j(\mathbf{y})) = m_j(\mathbf{y}) \quad \text{for all } s_i \text{ with } |s_i - z_j(\mathbf{y})| < \delta.$$

(c) *For the fixed node system* $\mathbf{y}$ *consider a corresponding cut of the torus . Let* $i_1, \dots, i_k \in \{0, \dots, n\}$ *be pairwise different, and suppose that* $\widehat{I}_{i_1}(\mathbf{y}), \dots, \widehat{I}_{i_k}(\mathbf{y})$ *are non-degenerate and* $\widehat{z}_{i_j} \in \operatorname{int} \widehat{I}_{i_j}$ *for each* $j = 1, \dots, k$. *Then there is* $\eta > 0$ *such that for all* $0 < h < \eta$

$$\widehat{m}_{i_j}(\mathbf{y} + h\mathbf{a}) < \widehat{m}_{i_j}(\mathbf{y}) \quad j = 1, \dots, k.$$

Lemma. *Let the kernel functions* $K_0, \ldots, K_n$ *be concave. Suppose that* $I_j(\mathbf{y}) = [y_j, y_{j'}]$ *is degenerate, i.e., a singleton.*

(a) *Suppose the kernels satisfy Condition* (∞'). *Then there exists* $\varepsilon > 0$ *such that for all* $0 < |t - y_j| < \varepsilon$ *we have* $F(\mathbf{y}, t) > F(\mathbf{y}, y_j)$.

(b) *Suppose the kernels are in* $\mathrm{C}^1(0, 2\pi)$ *and are non-constant. Then there exists* $\varepsilon > 0$ *such that either for all* $t \in (y_j - \varepsilon, y_j)$ *or for all* $t \in (y_j, y_j + \varepsilon)$ *we have* $F(\mathbf{y}, t) > F(\mathbf{y}, y_j)$.

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Corollary. *Let the kernel functions $K_0, \ldots, K_n$ be concave. Suppose that $I_j(\mathbf{y})$ is degenerate.*

(a) *Suppose the kernels satisfy Condition (∞'). Then for any neighboring, non-degenerate arc I_ℓ we have $m_\ell(\mathbf{y}) > m_j(\mathbf{y})$.*

(b) *Suppose the kernels are in $\mathrm{C}^1(0, 2\pi)$ and are non-constant. Then for at least one neighboring, non-degenerate arc I_ℓ we have $m_\ell(\mathbf{y}) > m_j(\mathbf{y})$.*

Lemma. *Let the kernel functions* $K_0, \ldots, K_n$ *be concave. Suppose that* $I_j(\mathbf{y}) = [y_j, y_{j'}]$ *is degenerate, i.e., a singleton.*

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Corollary. *Let the kernel functions* $K_0, \ldots, K_n$ *be concave. Suppose that* $I_j(\mathbf{y})$ *is degenerate.*

(a) *Suppose the kernels satisfy Condition* (∞'). *Then for any neighboring, non-degenerate arc* I_ℓ *we have* $m_\ell(\mathbf{y}) > m_j(\mathbf{y})$.

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Corollary. *If* $K_0, \ldots, K_n$ *are non-constant, concave kernel functions and either all satisfy* (∞') *or all belong to* $\mathrm{C}^1(0, 2\pi)$, *then an equioscillation point* $\mathbf{e} \in \mathbb{T}^n$ *must belong to the interior of some simplex* $S = S_\sigma$, *i.e., we have* $\mathbf{e} \in X = \bigcup_\sigma S_\sigma$.

Example. *It can happen that an equioscillation point falls on the boundary of a simplex* S*, and that maximum points of non-degenerate arcs lie on the endpoints. This occurs e.g. for* $K_0 := -4\pi^3/|x|$ *on* $[-\pi, \pi)$*, extended periodically, and letting* $K_1(x) := K_2(x) := -(x-\pi)^2$ *on* $(0, 2\pi)$*, again extended periodically.*

Example. *It can happen that an equioscillation point falls on the boundary of a simplex S, and that maximum points of non-degenerate arcs lie on the endpoints. This occurs e.g. for $K_0 := -4\pi^3/|x|$ on $[-\pi, \pi)$, extended periodically, and letting $K_1(x) := K_2(x) := -(x-\pi)^2$ on $(0, 2\pi)$, again extended periodically.*

Proposition. *Suppose the kernels $K_0, \dots, K_n$ are strictly concave and either all satisfy (∞') or all belong to $\mathrm{C}^1(0, 2\pi)$. Let $\mathbf{w}^* \in \mathbb{T}^n$ be a local minimum point of $\overline{m}$, i.e., such that for some $\eta > 0$*

$$\overline{m}(\mathbf{w}^*) = \min_{|\mathbf{y}-\mathbf{w}^*|<\eta} \overline{m}(\mathbf{y}).$$

Then $\mathbf{w}^*$ is an equioscillation point, i.e.,

$$m_j(\mathbf{w}^*) = \overline{m}(\mathbf{w}^*) \quad \textit{for all } j = 0, \dots, n.$$

As a consequence, such a local minimum point belongs to $X = \bigcup_\sigma S_\sigma$.

Corollary. *Suppose the kernels $K_0, \ldots, K_n$ are strictly concave, and either all satisfy* (∞')*, or all belong to* $\mathrm{C}^1(0, 2\pi)$*.*
Let $S = S_\sigma$ be a simplex, and let $\mathbf{w}^* \in \overline{S}$ be an extremal node system for (3)*. Then the following assertions hold.*

(a) *If $\mathbf{w}^* \in S$, then $\mathbf{w}^*$ is an equioscillation point.*

(b) *Even in case $\mathbf{w}^* \in \partial S$ we have that $\mathbf{w}^*$ is a weak equioscillation point.*

(c) *Furthermore, if also* (∞) *holds, then we have $\{m_0(\mathbf{w}^*), \ldots, m_n(\mathbf{w}^*)\} \subseteq \{-\infty, M(S)\}$, with $m_j(\mathbf{w}^*) = -\infty$ iff $I_j(\mathbf{y})$ is degenerate.*

(d) *If* (∞') *holds, and $\mathbf{w}^* \in \partial S$, then there exists another simplex $S' = S_\pi$ with $\mathbf{w}^* \in \overline{S} \cap \overline{S'}$ with $M(S') < M(S)$, moreover $\mathbf{w}^*$ is not even a local (conditional) minimum within $\overline{S'}$.*

Corollary. *Suppose the kernels $K_0, \dots, K_n$ are strictly concave, and either all satisfy* (∞')*, or all belong to $\mathrm{C}^1(0, 2\pi)$.*
Let $S = S_\sigma$ be a simplex, and let $\mathbf{w}^* \in \overline{S}$ be an extremal node system for (3). *Then the following assertions hold.*

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Corollary. *Suppose the kernels* $K_0, \dots, K_n$ *are strictly concave and either all satisfy* (∞')*, or all belong to* $\mathrm{C}^1(0, 2\pi)$*. If* $\mathbf{w}$ *is an extremal node system for* (1)*, i.e.,*

$$\overline{m}(\mathbf{w}) = \min_{\mathbf{y} \in \mathbb{T}^n} \overline{m}(\mathbf{y}) = M,$$

then the nodes w_j *(*$j = 0, \dots, n$*) are pairwise different (i.e.,* $\mathbf{w} \in X$*) and, moreover,* $\mathbf{w}$ *is an equioscillation point, i.e., we have*

$$m_j(\mathbf{w}) = M \quad \text{for } j = 0, \dots, n.$$

Lemma. *Suppose the kernels $K_0, \ldots, K_n$ are strictly concave. Let $S = S_\sigma$ be a simplex. Then $F(\mathbf{y}, s) : \mathbb{T}^n \times \mathbb{T} \to [-\infty, \infty)$ restricted to the convex open set*

$$\mathcal{D} := \mathcal{D}_{\sigma,i} := \big\{(\mathbf{y}, s) : \mathbf{y} \in S \text{ and } s \in \operatorname{int} I_i(\mathbf{y})\big\} \quad \text{is strictly concave.}$$

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Proof. First, note that the set $\mathcal{D} := \mathcal{D}_{\sigma,i}$ is a convex subset of $\mathbb{T}^{n+1}$. Indeed, let $(\mathbf{x}, r), (\mathbf{y}, s) \in \mathcal{D}$ and $t \in [0,1]$. Then $x_i < x_\ell$ and $y_i < y_\ell$ implies $tx_i + (1-t)y_i < tx_\ell + (1-t)y_\ell$, and $x_i < r < x_\ell$, $y_i < s < y_\ell$ entails also $tx_i + (1-t)y_i < tr + (1-t)s < tx_\ell + (1-t)y_\ell$.

Lemma. *Suppose the kernels* $K_0, \ldots, K_n$ *are strictly concave. Let* $S = S_\sigma$ *be a simplex. Then* $F(\mathbf{y}, s) : \mathbb{T}^n \times \mathbb{T} \to [-\infty, \infty)$ *restricted to the convex open set*

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$$F(t(\mathbf{x}, r) + (1-t)(\mathbf{y}, s)) = \sum_{\ell=0}^{n} K_\ell(tr + (1-t)s - (tx_i + (1-t)y_i))$$

$$\geq \sum_{\ell=0}^{n} tK_\ell(r - x_i) + (1-t)K_\ell(s - (1-t)y_i) = tF(\mathbf{x}, r) + (1-t)F(\mathbf{y}, s).$$

This shows concavity of F. To see strict concavity suppose $t \neq 0, 1$ and that $(\mathbf{x}, r), (\mathbf{y}, s) \in \mathcal{D}$ are different points. If $r \neq s$, then using strict concavity of K_0 we must have $K_0(tr + (1-t)s) > tK_0(r) + (1-t)K_0(s)$, and if $r = s$, but $x_\ell \neq y_\ell$ for some $1 \leq \ell \leq n$, then using strict concavity of K_ℓ (and also that $r = s$) it follows that $K_\ell(tr + (1-t)s - (tx_\ell + (1-t)y_\ell)) = K_\ell(s - (tx_\ell + (1-t)y_\ell)) > tK_\ell(s - x_\ell) + (1-t)K_\ell(s - y_\ell)$.

Proposition. *Suppose the kernels $K_0, \ldots, K_n$ are strictly concave. Then for all $i = 0, 1, \ldots, n$, the functions $m_i(\mathbf{y}) : S \to \mathbb{R}$ are also strictly concave. As a consequence,*

$$\underline{m} : S \to [-\infty, \infty), \quad \underline{m}(\mathbf{y}) := \min_{j=0,\ldots,n} m_j(\mathbf{y})$$

is a strictly concave function.

Proof. Take $\mathbf{x}, \mathbf{y} \in S$ and abbreviate $w_i := z_i(\mathbf{x})$, $v_i := z_i(\mathbf{y})$ (the unique maximum points of $F(\mathbf{x}, \cdot)$ and $F(\mathbf{y}, \cdot)$ in $I_i(\mathbf{x})$ and $I_i(\mathbf{y})$, respectively, i.e., $m_i(\mathbf{y}) = F(\mathbf{y}, v_i)$, $m_i(\mathbf{x}) = F(\mathbf{x}, w_i)$). Let $\zeta(t) := z_i(t\mathbf{x} + (1-t)\mathbf{y})$, $\zeta(0) = v_i$, $\zeta(1) = w_i$. According to the previous Lemma the function F is strictly concave, hence for different $\mathbf{x} \neq \mathbf{y}$ we necessarily have

$$F(t(\mathbf{x}, w_i) + (1-t)(\mathbf{y}, v_i)) > tF(\mathbf{x}, w_i)) + (1-t)F(\mathbf{y}, v_i) = tm_i(\mathbf{x}) + (1-t)m_i(\mathbf{y}).$$

Here the left hand side can be written as $F(t\mathbf{x} + (1-t)\mathbf{y}, \omega(t))$ with $\omega(t) = tw_i + (1-t)v_i \in I_i(t\mathbf{x} + (1-t)\mathbf{y})$. Thus by the definition of m_i we have

$$m_i(t\mathbf{x} + (1-t)\mathbf{y}) = \max_{s \in I_i(t\mathbf{x}+(1-t)\mathbf{y})} F(t\mathbf{x} + (1-t)\mathbf{y}, s) \geq F(t(\mathbf{x}, w_i) + (1-t)(\mathbf{y}, v_i)).$$

Hence, the previous considerations yield even $m_i(t\mathbf{x} + (1-t)\mathbf{y}) > tm_i(\mathbf{x}) + (1-t)m_i(\mathbf{y})$, whence the first assertion follows. Since minimum of strictly concave functions is strictly concave, the last assertion follows, too. □

Corollary. *Suppose the kernels $K_0, \ldots, K_n$ are strictly concave, and let $S := S_\sigma$ be a simplex.*

(a) *In $\overline{S}$ the function $\underline{m}$ has a* **unique** *global maximum point $\mathbf{y}_*$, and no local minimum point in S.*

(b) *If the kernels satisfy* (∞), *then $\mathbf{y}_* \in S$.*

(c) *There is no other point in $\overline{S}$ majorizing $\mathbf{y}_*$ than $\mathbf{y}_*$ itself.*

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Proof. (a) Since $\underline{m}$ is strictly concave on S and continuous on $\overline{S}$ the assertion is evident.

(b) Under condition (∞) we have $\underline{m}|_{\partial S} = -\infty$, whence the assertion follows.

(c) If $\mathbf{x} \in \overline{S}$ with $m_j(\mathbf{x}) \geq m_j(\mathbf{y}_*)$ for all $j = 0, 1, \ldots, n$, then for the minimum $\underline{m} := \min_{j=0,\ldots,n} m_j$ we also have $\underline{m}(\mathbf{x}) \geq \underline{m}(\mathbf{y}_*)$, whence $\mathbf{x}$ is also a maximum point, and by uniqueness (part (a)) this entails $\mathbf{x} = \mathbf{y}_*$. $\square$

Corollary. *Suppose the kernels $K_0, \dots, K_n$ are strictly concave.*

(a) *Let $\mathbf{y} \in S = S_\sigma$, $\mathbf{x} \in \overline{S}$, $\mathbf{x} \neq \mathbf{y}$ be such that $\mathbf{x}$ majorizes $\mathbf{y}$, i.e., $m_j(\mathbf{x}) \geq m_j(\mathbf{y})$ for each $j = 0, \dots, n$. Then there are $\mathbf{a} \in \mathbb{R}^n$ and $\delta > 0$ such that for every $j = 0, \dots, n$*

$$m_j(\mathbf{y} - t\mathbf{a}) < m_j(\mathbf{y}) \qquad \big(t \in (0, \delta)\big), \quad m_j(\mathbf{y} + t\mathbf{a}) > m_j(\mathbf{y}) \qquad \big(t \in (0, 1)\big).$$

In particular, the Local Strict Non-Majorization and Non-Minorization Properties fail at $\mathbf{y}$.

(b) *The Local Non-Majorization Property, the Local Non-Minorization Property, the Local Comparison Property and the Comparison Property are all equivalent, also together with their strict versions.*

(c) *If one has the Local Strict Non-Minorization Property at an* **interior** *point $\mathbf{y} \in S$, then also the Local Non-Majorization Property holds at the same point $\mathbf{y}$.*

Proof. (a) Take $\mathbf{a} := \mathbf{x} - \mathbf{y}$ and let

$$\mathbf{y}_t := \mathbf{y} + t\mathbf{a} = (1-t)\mathbf{y} + t\mathbf{x}.$$

For sufficiently small $\delta > 0$ we have $\mathbf{y}_t \in S$ for every $(-\delta, 1]$ (since S is convex and open). By the strict concavity of m_j we obtain for $t \in [0,1]$ that

$$m_j(\mathbf{y}_t) > (1-t)m_j(\mathbf{y}) + tm_j(\mathbf{x}) \geq (1-t)m_j(\mathbf{y}) + tm_j(\mathbf{y}) = m_j(\mathbf{y})$$

and for $t \in (-\delta, 0)$

$$m_j(\mathbf{y}_t) < (1-t)m_j(\mathbf{y}) + tm_j(\mathbf{x}) \leq (1-t)m_j(\mathbf{y}) + tm_j(\mathbf{y}) = m_j(\mathbf{y}).$$

This proves the first assertion.

(b) The Comparison Property evidently implies the Local Comparison Property and that implies further the Local Non-Minorization and Non-Majorization Properties. The already established first assertion (a) provides the converse implications even if we start with the even weaker Local Strict Non-Minorization or Non-Majorization Properties.

(c) Follows directly from (a) by contraposition. □

Proposition. *Suppose that the kernel functions $K_0, \ldots, K_n$ are strictly concave. Let $S = S_\sigma$ be a fixed simplex and let $\mathbf{e}, \mathbf{f} \in \overline{S}$ be two different equioscillation points.*

(a) *Then we have $M(S) < m(S)$, and the Sandwich Property fails.*

(b) *If $\overline{m}(\mathbf{e}) \leq \overline{m}(\mathbf{f})$ and $\mathbf{e} \in S$, then the Strict Local Non-Majorization and Non-Minorization Properties fail to hold at $\mathbf{e}$.*

(c) *If K_j either all satisfy (∞') or are all C^1, then the Comparison Property fails.*

Proposition. *Suppose that the kernel functions $K_0, \ldots, K_n$ are strictly concave. Let $S = S_\sigma$ be a fixed simplex and let $\mathbf{e}, \mathbf{f} \in \overline{S}$ be two different equioscillation points.*

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(c) *If K_j either all satisfy (∞') or are all C^1, then the Comparison Property fails.*

Proof. Assume, as we may, $\overline{m}(\mathbf{e}) \leq \overline{m}(\mathbf{f})$.

(a) If $\overline{m}(\mathbf{e}) < \overline{m}(\mathbf{f})$, then we obviously have $M(S) \leq \overline{m}(\mathbf{e}) < \overline{m}(\mathbf{f}) = \underline{m}(\mathbf{f}) \leq m(S)$. If, on the other hand, $\overline{m}(\mathbf{e}) = \overline{m}(\mathbf{f})$, then for the point $\mathbf{g} := \frac{1}{2}(\mathbf{e} + \mathbf{f}) \in \overline{S}$ by the strict concavity we find $m_j(\mathbf{g}) > \frac{1}{2}(m_j(\mathbf{e}) + m_j(\mathbf{f}) = \overline{m}(\mathbf{e})$ for all $j = 0, \ldots, n$, whence also $\underline{m}(\mathbf{g}) > \overline{m}(\mathbf{e})$ and thus also $m(S) \geq \underline{m}(\mathbf{g}) > \overline{m}(\mathbf{e}) \geq M(S)$. In both cases the Sandwich Property must fail, because a previous Remark this property is equivalent to $M(S) \geq m(S)$.

(b) If $\overline{m}(\mathbf{e}) \leq \overline{m}(\mathbf{f})$, then $\mathbf{f}$ majorizes $\mathbf{e}$, so a previous Corollary finishes the proof.

(c) The assertion follows from part (b), since equioscillation points belong to S. □

Corollary. *Suppose the kernels $K_0, \dots, K_n$ are strictly concave. Let $S := S_\sigma$ be a simplex and let $\mathbf{y}^* \in S$ be a local minimum of $\overline{m}$.*

(a) *Then there exists no other point $\mathbf{x}$, different from $\mathbf{y}^*$ in $\overline{S}$ majorizing $\mathbf{y}^*$.*

(b) *Suppose the kernels either all satisfy (∞') or are all C^1.*

Then in $\overline{S}$ there is no other local minimum point of $\overline{m}$ than $\mathbf{y}^*$.

In particular, $\overline{m}$ has a global minimum point at $\mathbf{y}^*$ in $\overline{S}$.

Corollary. *Suppose the kernels $K_0, \ldots, K_n$ are strictly concave. Let $S := S_\sigma$ be a simplex and let $\mathbf{y}^* \in S$ be a local minimum of $\overline{m}$.*

(a) *Then there exists no other point $\mathbf{x}$, different from $\mathbf{y}^*$ in $\overline{S}$ majorizing $\mathbf{y}^*$.*

(b) *Suppose the kernels either all satisfy (∞') or are all C^1.*

Then in $\overline{S}$ there is no other local minimum point of $\overline{m}$ than $\mathbf{y}^*$.

In particular, $\overline{m}$ has a global minimum point at $\mathbf{y}^*$ in $\overline{S}$.

Proof. (a) Suppose $\mathbf{x} \in \overline{S}$ majorizes $\mathbf{y}^*$ and $\mathbf{x} \neq \mathbf{y}$. Then there are $\mathbf{a} \in \mathbb{R}^n$ and $\delta > 0$ with $m_j(\mathbf{y}^* - t\mathbf{a}) < m_j(\mathbf{y}^*)$ for every $t \in (0, \delta)$ and $j = 0, \ldots, n$.
Hence $\mathbf{y}^*$ cannot be a local minimum point for $\overline{m}$.

(b) Under condition (∞') local minimum points of $\overline{m}$ are also equioscillation points, so at least one of two such points majorizes the other.
But then by part (a) the two point must be equal. □

Proposition. *Suppose the kernels $K_0, \ldots, K_n$ are strictly concave and either all satisfy (∞') or all are C^1. Let $S := S_\sigma$ be a simplex. If $\overline{m}$ has a local minimum point $\mathbf{y}^* \in S$, then $\mathbf{y}^*$ is a unique point of equioscillation in $\overline{S}$, and $\underline{m}$ has there its (unique, global) maximum. In particular, then $M(S) = m(S)$. Moreover, the Sandwich Property holds true in S. Furthermore, the Singular Non-Majorization and the Non-Minorization Properties hold on S.*

Proposition. *Suppose the kernels $K_0, \ldots, K_n$ are strictly concave and either all satisfy (∞') or all are C^1. Let $S := S_\sigma$ be a simplex. If $\overline{m}$ has a local minimum point $\mathbf{y}^* \in S$, then $\mathbf{y}^*$ is a unique point of equioscillation in $\overline{S}$, and $\underline{m}$ has there its (unique, global) maximum. In particular, then $M(S) = m(S)$. Moreover, the Sandwich Property holds true in S. Furthermore, the Singular Non-Majorization and the Non-Minorization Properties hold on S.*

Proof. Let $\mathbf{y}_* \in \overline{S}$ be the (unique, global) maximum point of $\underline{m}$. Obviously, $\min_{j=0,\ldots,n} m_j(\mathbf{y}_*) = \underline{m}(\mathbf{y}_*) \geq \underline{m}(\mathbf{y}^*)$. By assumption we conclude that $\mathbf{y}^*$ is an equioscillation point, i.e., $\underline{m}(\mathbf{y}^*) = \overline{m}(\mathbf{y}^*) = m_j(\mathbf{y}^*)$ for $j = 0, \ldots, n$. Thus we find that $\mathbf{y}_*$ majorizes the point $\mathbf{y}^*$. This is not possible unless $\mathbf{y}_* = \mathbf{y}^*$. Therefore we obtain $M(S) = m(S)$, and the Sandwich Property. If $\mathbf{e} \in \overline{S}$ is another equioscillation point, then $\overline{m}(\mathbf{e}) \geq \overline{m}(\mathbf{y}^*)$ (since $\mathbf{y}^*$ is a minimum point). This would imply $M(S) < m(S)$, which is nonsense. Since $\mathbf{y}^*$ is the unique equioscillation point in $\overline{S}$, by a previous Corollary there is no point majorizing it. But also $\mathbf{y}^*$ is the unique global minimum point of $\overline{m}$, so there is no point in $\overline{S}$ minorizing it. $\square$

Proposition. *Suppose that* $K_0, \dots, K_n$ *are* C^2 *with* $K_j'' < 0$ *(*$j = 0, \dots, n$*), and let* S *be a simplex. For* $j = 0, \dots, n$ *the functions* $m_j(\mathbf{y})$ *are continuously differentiable in* S *and*

$$\frac{\partial m_j}{\partial y_r}(\mathbf{y}) = -K_r'\big(z_j(\mathbf{y}) - y_r\big) \quad \text{for } r = 1, \dots, n.$$

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$$\frac{\partial m_j}{\partial y_r}(\mathbf{y}) = -K_r'\big(z_j(\mathbf{y}) - y_r\big) \quad \text{for } r = 1, \dots, n.$$

Proof. $\mathbf{y} \in S$ fixed. Recall that $t := z_j(\mathbf{y})$ is the unique maximum point in $I_j(\mathbf{y})$, hence $F'(\mathbf{y}, t) = 0$. Obviously

$$F''(\mathbf{y}, t) = K_0''(t) + \sum_{j=1}^{n} K_j''(t - y_j) < 0$$

so, by the implicit function theorem, $z_j(.)$ is continuously differentiable near $\mathbf{y}$. $m_j(\mathbf{y}) = F(\mathbf{y}, z_j(\mathbf{y}))$ implies that m_j is smooth too, and

$$\begin{aligned}\frac{\partial m_j}{\partial y_r}(\mathbf{y}) &= \frac{\partial}{\partial y_r}\Big(F\big(\mathbf{y}, z_j(\mathbf{y})\big)\Big) = \frac{\partial F}{\partial y_r}\big(\mathbf{y}, z_j(\mathbf{y})\big) + \frac{\partial}{\partial t}F\big(\mathbf{y}, t\big)\big|_{t=z_j(\mathbf{y})}\frac{\partial}{\partial y_r}z_j\big(\mathbf{y}\big) \\ &= -K_r'\big(z_j(\mathbf{y}) - y_r\big).\end{aligned}$$

□

For a given permutation σ of $\{1,\ldots,n\}$, consider the mapping Δ_σ defined by

$$\Delta_\sigma(\mathbf{y}) := (m_{\sigma(1)}(\mathbf{y}) - m_{\sigma(0)}(\mathbf{y}), \ldots, m_{\sigma(n)}(\mathbf{y}) - m_{\sigma(n-1)}(\mathbf{y}))^\top.$$

Its Jacobian matrix $D\Delta_\sigma$ is

$$D\Delta_\sigma(\mathbf{y}) = \begin{array}{c} \\ \\ j \\ \\ \end{array} \begin{pmatrix} & r & \\ & \vdots & \\ \cdots & -K'_r\big(z_{\sigma(j)}(\mathbf{y}) - y_r\big) + K'_r\big(z_{\sigma(j-1)}(\mathbf{y}) - y_r\big) & \cdots \\ & \vdots & \end{pmatrix}$$

where $r = 1,\ldots,n$ and $j = 1,\ldots,n$.

Proposition. *Suppose that for each $j = 0,\ldots,n$ the kernel K_j is C^2 with $K''_j < 0$. Let $S = S_\sigma$ be a simplex and let $\mathbf{y} \in S$. The Jacobian matrix of*

$$\Delta_\sigma(\mathbf{y}) = (m_{\sigma(1)}(\mathbf{y}) - m_{\sigma(0)}(\mathbf{y}), \ldots, m_{\sigma(n)}(\mathbf{y}) - m_{\sigma(n-1)}(\mathbf{y}))^\top.$$

is non-singular. That is, on S, we have the Difference Jacobi Property.

Sketch of the proof.

The Jacobian matrix $A := D\Delta_\sigma(.)$ has a special structure. First, the entries of $-A$ are non-negative on the diagonal: $-K_r'(z_r - y_r) + K_r'(z_{r-1} - y_r)$, $r = 1, \dots, n$. Since $z_{r-1} < y_r < z_r$, $z_r < 2\pi$ and $z_{r-1} > 0$, we obtain $z_{r-1} - y_r < 0 < z_r - y_r$ and $z_r - y_r < 2\pi + z_{r-1} - y_r$. Now using the 2π periodicity of K_r' and that K_r' is strictly monotone decreasing, we obtain $K_r'(z_{r-1} - y_r) < K_r'(z_r - y_r)$, that is, $-K_r'(z_r - y_r) + K_r'(z_{r-1} - y_r) < 0$.

The entries of $-A$ are non-positive off the diagonal: for $i < r$ we have $z_{i-1} < z_i \leq z_{r-1} < y_r$. Therefore, $-2\pi < z_{i-1} - y_r < z_i - y_r < 0$ and using that K_r' is strictly monotone decreasing and 2π periodic, we can write $-K_r'(z_i - y_r) + K_r'(z_{i-1} - y_r) > 0$. The case $i > r$ goes similarly.

Second, the column sums of $-A$ are strictly positive, i.e., with $\mathbf{x} = (1, 1, \dots, 1)^\top \in \mathbb{R}^n$, $-A^\top \mathbf{x}$ is a strictly positive vector: it is telescopic

$$\sum_{i=1}^{n} -K_r'(z_i - y_r) + K_r'(z_{i-1} - y_r) = -K_r'(z_n - y_r) + K_r'(z_0 - y_r).$$

Since $0 < z_0 < y_r < z_n < 2\pi$, we have $0 < z_n - y_r < 2\pi + z_0 - y_r < 2\pi$. Since K_r' is strictly decreasing and 2π periodic, it follows $-K_r'(z_n - y_r) + K_r'(z_0 - y_r) < 0$.

Sketch of the proof (cont'd).

The first property is sometimes expressed as $-A$ is a so-called Z-matrix, the together with the second property, it is equivalent to that $-A$ is a so-called M-matrix. M-matrices have lots of equivalent descriptions. These two properties imply that $-A$ is nonsingular. $\square$

Corollary. *Suppose that for each $j = 0, \ldots, n$ the kernel K_j is C^2 with $K_j'' < 0$ and satisfies* (∞)*. Let $S = S_\sigma$ be a simplex. Then $\Delta_\sigma : S \to \mathbb{R}^n$ is a homeomorphism.*

Corollary. *Suppose that for each $j = 0, \ldots, n$ the kernel K_j is C^2 with $K_j'' < 0$ and satisfies* (∞)*. Then all equioscillation points belong to some (open) simplex, and in each simplex $S = S_\sigma$ there is a unique equioscillation point.*

Lemma. *Suppose that $K_0, \dots, K_n$ are strictly concave kernel functions and a sequence of strictly concave kernel functions $(K_j^{(k)})_{k=1}^{\infty}$ converges uniformly (in the extended sense) to K_j as $k \to \infty$. Let $\mathbf{e}^{(k)} \in S$ be equioscillation points for the system of kernels $K_j^{(k)}$, $j = 0, \dots, n$. Then any accumulation point $\mathbf{e} \in \overline{S}$ of the sequence $(\mathbf{e}^{(k)}_{k\in\mathbb{N}}$ is an equioscillation point of the system K_j, $j = 0, \dots, n$.*

Lemma. *Let $f : [0, 1) \to \mathbb{R}$ be a strictly convex, increasing function. Then to each $\varepsilon > 0$ there exists another strictly convex increasing function $g : [0, 1) \to \mathbb{R}$ such that $g \in C^2[0, 1)$, $g'' > 0$ on $[0, 1)$, and $f(x) \le g(x) \le f(x) + \varepsilon$ all over $[0, 1)$.*

Theorem. *Suppose that for each $j = 0, \dots, n$ the kernels K_j are strictly concave. Then for each simplex $S = S_\sigma$ there exists an equioscillation point in $\overline{S}$. Moreover, if the kernels are either all in $C^1(0, 2\pi)$ or all satisfy (∞'), then any equioscillation point is in the open simplex S.*

Corollary. *Let the kernel functions $K_0, \dots, K_n$ be strictly concave. Then in any simplex $S = S_\sigma$ the Equioscillation Property holds, and we have $M(S) \le m(S)$.*

Theorem. *Suppose the kernel functions $K_0, K_1, \dots, K_n$ are strictly concave and either all satisfy (∞') or all are C^1. Then there is $\mathbf{w}^* \in \mathbb{T}^n$, $\mathbf{w}^* = (w_1, \dots, w_n)$ with*

$$M := \inf_{\mathbf{y} \in \mathbb{T}^n} \sup_{t \in \mathbb{T}} F(\mathbf{y}, t) = \sup_{t \in \mathbb{T}} F(\mathbf{w}^*, t).$$

Moreover, we have the following:

(a) $\mathbf{w}^*$ *is an equioscillation point, i.e.,* $m_0(\mathbf{w}^*) = \cdots = m_n(\mathbf{w}^*)$.

(b) $\mathbf{w}^* \in S := S_\sigma$ *for some simplex, i.e., the nodes in* $\mathbf{w}^*$ *are different, and*

$$\inf_{\mathbf{y} \in S} \max_{j=0,\dots,n} \sup_{t \in I_j(\mathbf{y})} F(\mathbf{y}, t) = M = \sup_{\mathbf{y} \in S} \min_{j=0,\dots,n} \sup_{t \in I_j(\mathbf{y})} F(\mathbf{y}, t).$$

(c) *We have the Sandwich Property on* S, *i.e., for each* $\mathbf{x}, \mathbf{y} \in S$

$$\underline{m}(\mathbf{x}) \le M \le \overline{m}(\mathbf{y}).$$

Example.

$$K(x) := \pi - |x - \pi| \qquad \text{for } x \in [0, 2\pi]$$

$$Q(x) := |x|(2\pi - |x|) \qquad \text{for } x \in [0, 2\pi],$$

and extend them periodically to $\mathbb{R}$. Take $K_0 = K_1 = K$ and $K_2 = K_3 = \varepsilon Q$ where $\varepsilon > 0$ will be given later. Consider the node system $y_0 = 0$, $y_1 = \pi$, $y_2 = \frac{\pi}{2}$, $y_3 = \frac{3\pi}{2}$, $\mathbf{y} = (y_1, y_2, y_3, y_4)$. We can write

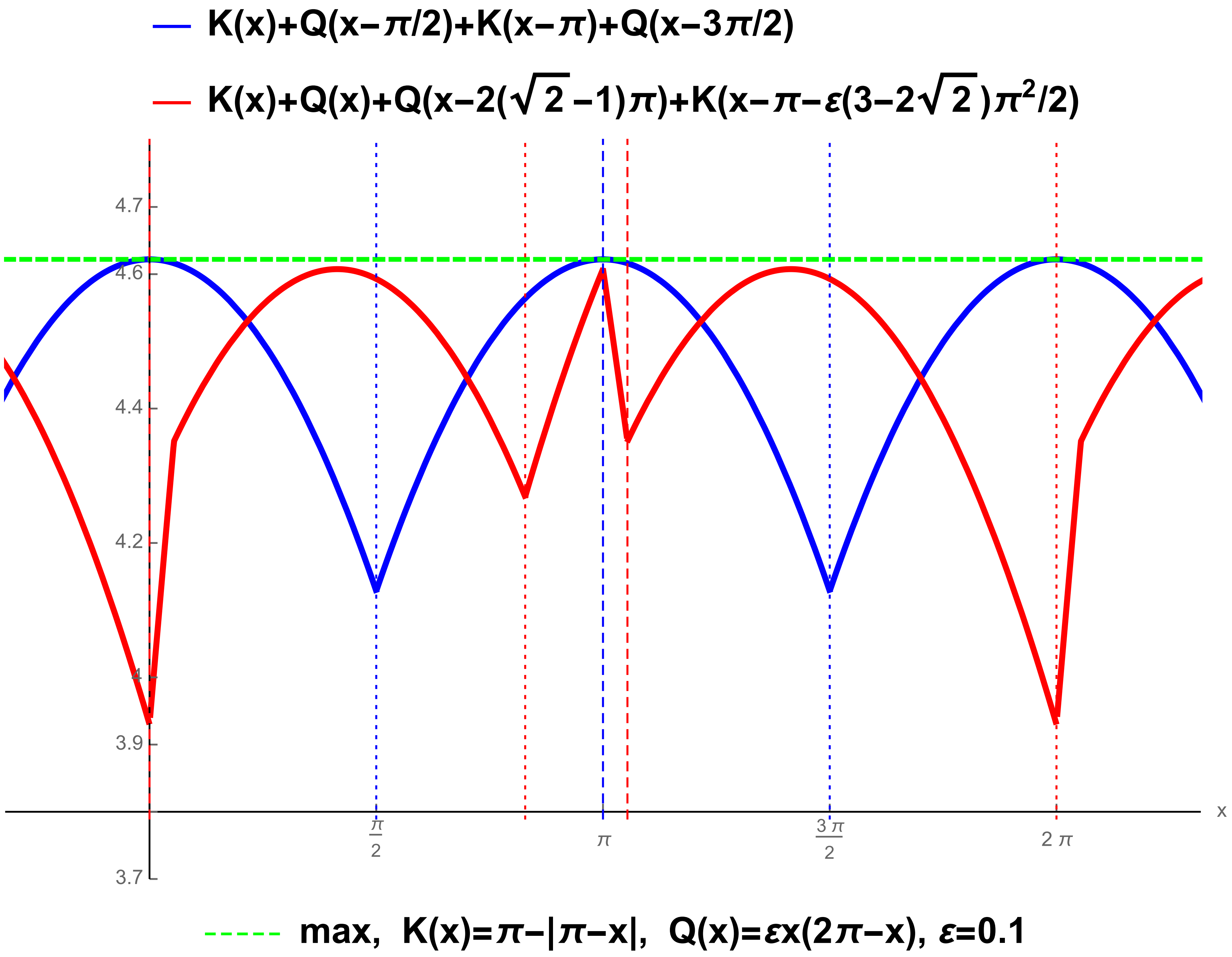
$$F(\mathbf{y}, t) = K_0(t) + K_1(t - y_1) + K_2(t - y_2) + K_3(t - y_3) = \pi + \varepsilon Q(t - \tfrac{\pi}{2}) + \varepsilon Q(t - \tfrac{3\pi}{2}).$$

Then $m_0(\mathbf{y}) = F(\mathbf{y}, 0) = \max_{t \in [0, \frac{\pi}{2}]} F(\mathbf{y}, t) = \pi + 3\varepsilon \frac{\pi^2}{2}$ and by symmetry $m_0(\mathbf{y}) = m_1(\mathbf{y}) = m_2(\mathbf{y}) = m_3(\mathbf{y})$, i.e., $\mathbf{y}$ is an equioscillation point.

Put $w_0 = 0$, $w_1 = \pi + (3 - 2\sqrt{2})\varepsilon\pi^2$, $w_2 = (2\sqrt{2} - 2)\pi$, $w_3 = 2\pi$, $\mathbf{w} = (w_1, w_2, w_3, w_4)$.

Then $m_0(\mathbf{w}) = \pi + \varepsilon\pi^2(6\sqrt{2} - 7)$, $m_1(\mathbf{w}) = \varepsilon\pi^2(6\sqrt{2} - 7)$, $m_2(\mathbf{w}) = \pi + \varepsilon\pi^2(6\sqrt{2} - 7)$, $m_3(\mathbf{w}) = \pi + \varepsilon\pi^2(14\sqrt{2} - 19)$ and $\overline{m}(\mathbf{w}) = m_3(\mathbf{w})$. Hence

$$\overline{m}(\mathbf{y}) = \varepsilon\pi^2 \frac{3}{2} > \varepsilon\pi^2(6\sqrt{2} - 7) = \overline{m}(\mathbf{w}).$$


K(x)+Q(x−π/2)+K(x−π)+Q(x−3π/2)
K(x)+Q(x)+Q(x−2(√2−1)π)+K(x−π−ε(3−2√2)π²/2)
4.7
4.6
4.4
4.2
4
3.9
3.7
π/2
π
3π/2
2π
x
max, K(x)=π−|π−x|, Q(x)=εx(2π−x), ε=0.1

Example.

The kernels in the previous example can be approximated with "nice" kernels using the convergence theorems. Consider the kernel functions K_0, K_1, K_2, K_3 from previous Example, and let $K_0^{(k)}$, $K_1^{(k)}$, $K_2^{(k)}$, $K_3^{(k)}$ be strictly concave, symmetric, satisfying the condition (∞') and

$$K_j^{(k)} \to K_j \quad \text{uniformly as } k \to \infty \text{ for } j = 0, 1, 2, 3.$$

By an earlier Proposition, we have $M^{(k)}(S) \to M^{(k)}(S)$ for any simplex. So for sufficiently large $k \in \mathbb{N}$ we must have

$$M^{(k)}(S_\sigma) \neq M^{(k)}(S_{\sigma'}) \tag{8}$$

for the two different simplexes corresponding to the permutations $\sigma = (213)$ and $\sigma' = (123)$.

Example.

Let K_0, K_1, K_2, K_3 be kernel functions satisfying (∞') with

$$M^{(k)}(S_\sigma) \neq M^{(k)}(S_{\sigma'})$$

for different simplexes S_σ and $S_{\sigma'}$ (see the preceding example). Let $K_0^{(k)}$, $K_1^{(k)}$, $K_2^{(k)}$, $K_3^{(k)}$ be C^2, strictly concave, symmetric and satisfying the condition (∞) such that

$$K_j^{(k)} \to K_j \quad \text{locally uniformly on } (0, 2\pi)$$

as $k \to \infty$ for $j = 0, 1, 2, 3$ (using an earlier Lemma). We know that $M^{(k)}(S) \to M^{(k)}(S)$ for any simplex. So for sufficiently large $k \in \mathbb{N}$ we must have

$$M^{(k)}(S_\sigma) \neq M^{(k)}(S_{\sigma'}).$$

Theorem. *Suppose the kernel functions L, K are strictly concave and either both satisfy (∞') or are in C^1. Set $F(\mathbf{y},t) := L(t) + \sum_{j=1}^n K(t-y_j)$. Then there is a unique $\mathbf{w}^* \in \mathbb{T}^n$, $\mathbf{w}^* = (w_1, \ldots, w_n)$ with*

$$M := \inf_{\mathbf{y}\in\mathbb{T}^n} \sup_{t\in\mathbb{T}} F(\mathbf{y},t) = \sup_{t\in\mathbb{T}} F(\mathbf{w}^*,t).$$

Moreover, we have the following:

(a) $\mathbf{w}^*$ is an equioscillation point, i.e., $m_0(\mathbf{w}^*) = \cdots = m_n(\mathbf{w}^*)$.

(b) The nodes $w_0, \ldots, w_n$ are different and $\mathbf{w}^*$ is an equioscillation point, i.e.,

$$m_0(\mathbf{w}^*) = \cdots = m_n(\mathbf{w}^*).$$

(c) We have

$$\inf_{\mathbf{y}\in\mathbb{T}^n} \max_{j=0,\ldots,n} \sup_{t\in I_j(\mathbf{y})} F(\mathbf{y},t) = M = \sup_{\mathbf{y}\in\mathbb{T}^n} \min_{j=0,\ldots,n} \sup_{t\in I_j(\mathbf{y})} F(\mathbf{y},t).$$

(d) We have the Sandwich Property on $\mathbb{T}$, i.e., for each $\mathbf{x}, \mathbf{y} \in \mathbb{T}$

$$\underline{m}(\mathbf{x}) \le M \le \overline{m}(\mathbf{y}).$$

(e) If $K = L$, then the points $w_0, \ldots, w_n$ lie equidistantly in $\mathbb{T}$.

Lemma. *Let K be strictly concave an let $a, b > 0$, $x \in (0, 2\pi)$ be given. Then for each $x \in (0, 2\pi)$ for sufficiently small $\delta > 0$ we have that*

$$\frac{1}{a}K(t-(x+ah)) + \frac{1}{b}K(t-(x-bh)) < \frac{1}{a}K(t-x) + \frac{1}{b}K(t-x)$$

for each $t \in (0, x-b\delta) \cup (x+a\delta, 2\pi)$ and each $0 < h < \delta$.

Lemma. *Let K be strictly concave an let $a, b > 0$, $x \in (0, 2\pi)$ be given. Then for each $x \in (0, 2\pi)$ for sufficiently small $\delta > 0$ we have that*

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for each $t \in (0, x-b\delta) \cup (x+a\delta, 2\pi)$ and each $0 < h < \delta$.

Proof. Let $\delta > 0$ be so small that for $h \in (0, \delta)$ we have $x - bh > 0$ and $x + ah < 2\pi$. Then $K(\cdot - x)$ is strictly concave on the intervals $(0, x)$ and $(x, 2\pi)$. By strict concavity the difference quotients are strictly decreasing in both variables, so that for all $h \in (0, \delta)$ and $t \in (0, x - b\delta)$ and $t \in (x + a\delta, 2\pi)$

$$\frac{K(t-x) - K(t-(x-bh))}{-bh} < \frac{K(t-x) - K(t-(x+ah))}{ah}.$$

But this inequality is equivalent to the assertion.

The next result generalizes that of Hardin, Kendall and Saff in that extent that we do not assume the kernels to be symmetric about 0. □

Corollary. *Let K be any concave kernel function, and let $0 = e_0, e_1, \ldots, e_n$ be the equidistant node system in $\mathbb{T}$. Consider*

$$F(\mathbf{y}, t) := K(t) + \sum_{j=1}^{n} K(t - y_j).$$

(a) *For $\mathbf{e} = (e_0, e_1, \ldots, e_n)$ we have*

$$\max_{t \in \mathbb{T}} F(\mathbf{e}, t) = M = \inf_{\mathbf{y} \in \mathbb{T}^{n+1}} \max_{t \in \mathbb{T}} F(\mathbf{y}, t),$$

i.e., $\mathbf{e}$ is a minimum point of $\overline{m}$. Moreover,

$$\inf_{\mathbf{y} \in \mathbb{T}^{n+1}} \max_{j=0,\ldots,n} m_j(\mathbf{y}) = M = m = \sup_{\mathbf{y} \in \mathbb{T}^{n+1}} \min_{j=0,\ldots,n} m_j(\mathbf{y}).$$

(b) *If K is strictly concave, then $\mathbf{e}$ is the unique (up to permutation of the coordinates) maximum point of $\underline{m}$ and the unique minimum point of $\overline{m}$.*

Proof. Since the permutation of the nodes is irrelevant we may restrict the consideration to the simplex $S := S_{\mathrm{id}}$, where id is the identical permutation.

Proof. Since the permutation of the nodes is irrelevant we may restrict the consideration to the simplex $S := S_{\mathrm{id}}$, where id is the identical permutation.

(a) Approximate K uniformly by kernel functions $K^{(k)}$ satisfying (∞'). By one of the main theorems, $M^{(k)} = \overline{m}^{(k)}(\mathbf{e})$ and $M^{(k)} = m^{(k)}$ and obviously $M^{(k)} = M^{(k)}(S)$, $m^{(k)} = m^{(k)}(S)$. Then we have $M^{(k)}(S) \to M(S) = M$, $m^{(k)}(S) \to m(S)$, $\underline{m}^{(k)}(\mathbf{e}) \to \underline{m}(\mathbf{e})$ and $\overline{m}^{(k)}(\mathbf{e}) \to \overline{m}(\mathbf{e})$. So that $\overline{m}(\mathbf{e}) = M = m(S) = M(S)$.

Proof. Since the permutation of the nodes is irrelevant we may restrict the consideration to the simplex $S := S_{\mathrm{id}}$, where id is the identical permutation.

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(b) Let $\mathbf{w}^* \in \overline{S}$ be minimum point of $\overline{m}$. By an application of the foregoing with $a = b = 1$, we can conclude that $\mathbf{w}^* \in S$. Since $\mathbf{e} \in S$ is an equioscillation point and $\overline{m}(\mathbf{e}) \geq M(S)$, we see that $\mathbf{e}$ majorizes $\mathbf{w}^*$. But this is impossible for $\mathbf{e} \neq \mathbf{w}^*$. $\square$

Now our goal is to generalize Bojanov's result.

Based on the product representation of trigonometric polynomials, the following class of functions is called generalized trigonometric polynomials (GTP)

$$a\prod_{j=1}^{m}\left|\sin\frac{t-z_j}{2}\right|^{r_j} \text{ where } a, r_j > 0,\ z_j \in \mathbb{C}$$

for all $j = 1, \ldots, m$. When r_j's are positive integers, these are absolute values of trigonometric polynomials.

In the next theorem, we describe Chebyshev type (having minimal sup norm and fixed leading coefficient) GTPs when the multiplicities of the zeros are fixed and the zeros are real.

Theorem. *Let* $r_0, r_1, \ldots, r_n$ *be fixed positive real numbers. Then, there exists a unique system of points* $\mathbf{w}^* = (w_0, w_1, \ldots, w_n)$, $0 = w_0 < w_1 < \ldots < w_n < 2\pi$ *such that*

$$\left\| \left|\sin \frac{t-w_0}{2}\right|^{r_0} \ldots \left|\sin \frac{t-w_n}{2}\right|^{r_n} \right\| = \inf_{0=y_0\le y_1<\ldots<y_n<2\pi} \left\| \left|\sin \frac{t-y_0}{2}\right|^{r_0} \ldots \left|\sin \frac{t-y_n}{2}\right|^{r_n} \right\|$$

where $\|.\|$ *denotes the sup norm over* $[0,2\pi]$. *The extremal* $T^*(t) := \left|\sin \frac{t-w_0}{2}\right|^{r_0} \ldots \left|\sin \frac{t-w_n}{2}\right|^{r_n}$ *has the properties that there exists* $0 < z_0 < z_1 < z_2 < \ldots < z_n < 2\pi$ *such that* w_j*'s and* z_j*'s interlace (i.e.* $0 = w_0 < z_0 < w_1 < \ldots < w_n < z_n < w_0 + 2\pi = 2\pi$*) and* $T^*(z_j) = \|T^*\|$ *(*$j = 0, 1, \ldots, n$*).*

The proof is based on that if $T(t) = \prod_{j=0}^{n} \left|\sin \frac{t-y_j}{2}\right|^{r_j}$ and using $K(x) := \log|\sin(x/2)|$ as (C^2 smooth, strictly concave) kernel function, we can apply our main theorem.

The interval case:

$r_1, r_2, \ldots, r_n > 0$, and consider $|x - x_1|^{r_1} \ldots |x - x_n|^{r_n}$. Such functions are sometimes called generalized algebraic polynomials (GAP). Fix $[a, b] \subset \mathbb{R}$ and consider the following problem

$$\inf_{a \le x_1 < \ldots < x_n \le b} \left\| |x - x_1|^{r_1} \ldots |x - x_n|^{r_n} \right\| \tag{9}$$

where $\|.\|$ denotes the sup norm over $[a, b]$.

To tackle this case, let us first investigate the problem

$$\inf \left\| \left|\sin \frac{t - y_1}{2}\right|^{r_n} \ldots \left|\sin \frac{t - y_n}{2}\right|^{r_1} \left|\sin \frac{t - y_{n+1}}{2}\right|^{r_1} \ldots \left|\sin \frac{t - y_{2n}}{2}\right|^{r_n} \right\| \tag{10}$$

where the infimum is taken for $0 \le y_1 < \ldots < y_n < y_{n+1} < \ldots < y_{2n} < 2\pi$.

Theorem. *Using the notations introduced above, (10) has unique solution* $\mathbf{w}^* = (w_1, w_2, \ldots, w_{2n})$ *with* $w_1 + (w_{2n} - 2\pi) = 0$ *and* $0 < w_1 < \ldots < w_{2n} < \pi$*. Further,* $\mathbf{w}^*$ *is symmetric:* $w_k = 2\pi - w_{2n+1-k}$ *(*$k = 1, 2, \ldots, n$*).*

This theorem follows from the next more general, symmetry theorem.

Theorem. *Let* $K_1, \ldots, K_n$ *be strictly concave kernels such that* K_j *is even:* $K_j(t) = K_j(-t)$ *for all* $j = 1, \ldots, n$*. Take the simplex* $S_+ := \{0 \le y_1 < y_2 < \ldots < y_{2n} < 2\pi\}$*. Define the symmetric potential*

$$F_{symm}(\mathbf{y}, t) := K_1(t - y_1) + \ldots + K_{n-1}(t - y_{n-1}) + K_n(t - y_n) + K_n(t - y_{n+1}) + K_{n-1}(t - y_{n+2}) + \ldots + K_1(t - y_{2n}) \quad (11)$$

and consider the "doubled" trigonometric problem

$$M_{symm} := \inf_{\mathbf{y} \in S_+} \sup_{t \in [0, 2\pi)} F_{symm}(\mathbf{y}, t). \quad (12)$$

Then there is a unique solution $\mathbf{w}^* = (w_1, w_2, \ldots, w_{2n}) \in S_+$ *with* $w_1 + (w_{2n} - 2\pi) = 0$*. Further,* $\mathbf{w}^*$ *is symmetric:* $w_k = 2\pi - w_{2n+1-k}$ *(*$k = 1, 2, \ldots, n$*) and there are exactly* $2n$ *points:* $0 = z_1 < z_2 < \ldots < z_{n+1} = \pi < \ldots < z_{2n}$ *where* $F_{symm}(\mathbf{w}^*, .)$ *attains its supremum. Moreover,* z_j*'s and* w_j*'s interlace and* z_j*'s are symmetric too:* $z_k = 2\pi - z_{2n+1-k}$ *(*$k = 1, 2, \ldots, n$*).*

Proof.

Following the symmetric definition, we denote $K_{n+k}(t) := K_{n+1-k}(-t)$ where $k = 1, 2, \dots, n$, and by symmetry,

$$K_{n+k}(t) = K_{n+1-k}(t) \text{ for } k = -n+1, \dots, n. \tag{13}$$

Hence $F_{symm}(\mathbf{y}, t) = \sum_{j=1}^{2n} K_j(t - y_j)$.

The existence and uniqueness follow from our main theorem. That is, there exists a unique $\mathbf{w}^* = (w_1, w_2, \dots, w_{2n}) \in S_+$ with the normalization $w_1 = 0$ such that $M(S_+) = \overline{m}(\mathbf{w}^*)$. Furthermore, $M(S) = m(S)$ and $F(\mathbf{w}^*; .)$ equioscillates, hence $m(S_+) = \underline{m}(\mathbf{w}^*)$. Using rotation, we can assume that $w_1 > 0$ is such that $w_1 + (w_{2n} - 2\pi) = 0$. Now we establish $w_k = 2\pi - w_{2n+1-k}$ ($k = 1, 2, \dots, n$). By the assumption, it holds for $k = 1$, i.e. $w_1 = 2\pi - w_{2n}$. Reflect the w_k's: $v_k := 2\pi - w_{2n+1-k}$, $k = 1, \dots, 2n$ and write $\mathbf{v} := (v_1, \dots, v_{2n})$. Then $v_1 = w_1$ and $v_{2n} = w_{2n}$. Furthermore, put $L_k(t) := K_{2n+1-k}(-t)$ and consider

$$\tilde{F}(\mathbf{v}, t) := \sum_{k=-n+1}^{n} L_{n+k}(t - v_{n+k})$$

the potential of the reflected configuration.

We can write, using (13) and the even property of kernels,

$$\begin{aligned} L_{n+k}\left(t - v_{n+k}\right) &= K_{n+1-k}\left(-t + v_{n+k}\right) = K_{n+1-k}\left(t - v_{n+k}\right) \\ &= K_{n+1-k}\left(t - 2\pi + w_{n+1-k}\right) = K_{n+1-k}\left((2\pi - t) - w_{n+1-k}\right) \end{aligned}$$

for all $k = -n+1, \ldots, n$. Hence

$$\begin{aligned} \tilde{F}(\mathbf{v}, t) = \sum_{k=-n+1}^{n} L_{n+k}\left(t - v_{n+k}\right) &= \sum_{k=-n+1}^{n} K_{n+1-k}\left((2\pi - t) - w_{n+1-k}\right) \\ &= F_{symm}\left(\mathbf{w}^*; 2\pi - t\right) = F_{symm}\left(\mathbf{w}^*; -t\right). \end{aligned}$$

Obviously $\mathbf{v} \in S_+$. By definition, $m_j(\mathbf{w}^*) = \sup\{F_{symm}\left(\mathbf{w}^*; t\right) : w_j \leq t \leq w_{j+1}\}$, $j = 1, 2, \ldots, 2n-1$ and $m_0(\mathbf{w}^*) = m_{2n}(\mathbf{w}^*) = \sup\{F_{symm}\left(\mathbf{w}^*; t\right) : w_{2n} - 2\pi \leq t \leq w_1\}$ and similarly for $\mathbf{v}$, $m_j(\mathbf{v}) = \sup\{\tilde{F}\left(\mathbf{v}; t\right) : v_j \leq t \leq v_{j+1}\}$, $j = 1, 2, \ldots, 2n-1$ and $m_0(\mathbf{v}) = m_{2n}(\mathbf{v}) = \sup\{\tilde{F}\left(\mathbf{v}; t\right) : v_{2n} - 2\pi \leq t \leq v_1\}$.

Hence, we also have for $j = 1, 2, \ldots, 2n-1$

$$\begin{aligned} m_j(\mathbf{w}^*) &= \sup\left\{F_{symm}\left(\mathbf{w}^*; t\right) : w_j \le t \le w_{j+1}\right\} \\ &= \sup\left\{F_{symm}\left(\mathbf{w}^*; -t\right) : -w_{j+1} \le t \le -w_j\right\} = \sup\left\{\tilde{F}(\mathbf{v}, t) : -w_{j+1} \le t \le -w_j\right\} \\ &= \sup\left\{\tilde{F}(\mathbf{v}, t) : 2\pi - w_{j+1} \le t \le 2\pi - w_j\right\} = \sup\left\{\tilde{F}(\mathbf{v}, t) : v_{2n-j} \le t \le v_{2n+1-j}\right\} \\ &= m_{2n-j}(\mathbf{v}) \end{aligned}$$

and obviously $m_0(\mathbf{v}) = m_{2n}(\mathbf{v}) = m_0(\mathbf{w}^*) = m_{2n}(\mathbf{w}^*)$. This implies that $\underline{m}(\mathbf{v}) = \underline{m}(\mathbf{w}^*)$. Indirectly, suppose $\mathbf{v} \neq \mathbf{w}^*$. We use the strict concaveness of $\underline{m}\left(.\right)$ hence there is an $\mathbf{a} = (a_1, \ldots, a_{2n}) \in S_+$ such that $\underline{m}\left(\mathbf{a}\right) > \underline{m}\left(\mathbf{w}^*\right)$. But this contradicts that $\underline{m}\left(\mathbf{w}^*\right) = m\left(S_+\right) = \sup_{\mathbf{y} \in S_+} \underline{m}\left(\mathbf{y}\right)$. Therefore $\mathbf{v} = \mathbf{w}^*$, hence $w_k = 2\pi - w_{2n+1-k}$ $(k = 1, 2, \ldots, n)$.

The symmetry of w_k's implies the remaining assertions (interlacing and symmetry of z_j's).

We connect the interval problem (9) and the "doubled" trigonometric problem (10) using a classical idea transferring with $x = \cos(t)$ in the following (e.g. Szegő,1964).

Theorem. *We use the notations introduced above. Consider the algebraic problem* (9) *and the associated "doubled" trigonometric problem* (10)*. Denote the unique solution of* (10) *by* $\mathbf{w}^* = (w_1, \ldots, w_{2n})$ *and that of* (9) *by* $\mathbf{x} = (x_1, \ldots, x_n)$*, and let* $L(x) := \frac{b-a}{2}x + \frac{b+a}{2}$*.*

Then we can obtain $\mathbf{x}$ *from* $\mathbf{w}^*$ *:* $x_j = L(\cos w_{n+1-j})$*,* $j = 1, \ldots, n$*.*

Proof. For simplicity, assume that $a = -1$, $b = 1$, hence $L(x) = x$. We also use

$$\sin\frac{t-\alpha}{2}\sin\frac{t+\alpha-2\pi}{2} = \frac{1}{2}(\cos(t)\text{-}\cos(\alpha)) \tag{14}$$

moreover

$$\left|\sin\frac{t-t_1}{2}\right|^{r_n}\ldots\left|\sin\frac{t-t_n}{2}\right|^{r_1}\left|\sin\frac{t+t_n-2\pi}{2}\right|^{r_1}\ldots\left|\sin\frac{t+t_1-2\pi}{2}\right|^{r_n}$$
$$= \frac{1}{2^{\sum_{j=1}^{n} r_j}}\left|\cos(t)-\cos(t_1)\right|^{r_n}\ldots\left|\cos(t)-\cos(t_n)\right|^{r_1} \tag{15}$$

Therefore, for every GAP $P(x) = |x - x_1|^{r_1} \ldots |x - x_n|^{r_n}$ there is a GTP $T(t)$ of the form as in (10) such that $P(\cos t) = 2^{-\sum_{j=1}^{n} r_j} T(t)$. Now consider the problem (10). According to the symmetry theorem, there is a unique GTP $T^*(t)$ with minimal sup norm and

$$T^*(t) = \left|\sin \frac{t - w_1}{2}\right|^{r_n} \ldots \left|\sin \frac{t - w_n}{2}\right|^{r_1} \left|\sin \frac{t + w_n - 2\pi}{2}\right|^{r_1} \ldots \left|\sin \frac{t + w_1 - 2\pi}{2}\right|^{r_n}.$$

In view of (15), $P^*(x) := 2^{\sum_{j=1}^{n} r_j} T^*(\arccos x) = |x - \cos(w_1)|^{r_n} \ldots |x - \cos(w_n)|^{r_1}$ is the unique solution of the algebraic problem (9) and we can also write $P^*(x) = |x - x_1|^{r_1} \ldots |x - x_n|^{r_n}$ where $x_j = \cos w_{n+1-j}$, $j = 1, 2, \ldots, n$, hence $-1 \leq x_1 < \ldots < x_n \leq 1$.

Thank you for your attention!