1° a) $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} = \int_{x^2+y^2\le 1} \frac{2}{(1+x^2+y^2)^2}\,d(x,y) = \int_{0\,(r)}^{1}\int_{0\,(\alpha)}^{2\pi} \frac{2r}{(1+r^2)^2}\,d\alpha\,dr = 2\pi\left[\frac{-1}{1+r^2}\right]_0^1 = 2\pi\left(1-\frac{1}{2}\right) = \pi$

b) Ha $a_n \ge 0\ \forall n$, akkor $S_n$ monoton nő, ezért $\sum a_n$ konv $\Leftrightarrow$ $S_n$ korlátos

2° $\oint_L f\,dr = \int_T (f'_{2x} - f'_{1y})\,dT = \int_{-1}^{1}\int_{-1}^{1} (-x^2)\,dy\,dx = -2\int_{-1}^{1} x^2\,dx = -2\left[\frac{x^3}{3}\right]_{-1}^{1} = -\frac{4}{3}$

3° a) $\Phi'_x = 2xy - z \Rightarrow \Phi = x^2y - xz + c(y,z) \Rightarrow \Phi'_y = x^2 + c'_y = x^2 + z \Rightarrow \Phi = x^2y - xz + yz + d(z)$

$\Phi'_z = -x + y + d'(z) = -x + y \Rightarrow d = \text{const} \Rightarrow \Phi = x^2y - xz + yz + \text{const}$

b) Akárhány nyílt halmaz uniója nyílt, véges sok nyílt halmaz metszete nyílt.

4° $r(x,y) = (x, y, \sqrt{1-y^2})$, $0 \le x \le 1$, $-1 \le y \le 1$, $r'_x \times r'_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & -\frac{y}{\sqrt{1-y^2}} \end{vmatrix} =$

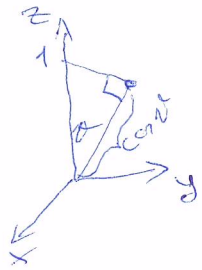
$= \left(0, \frac{y}{\sqrt{1-y^2}}, 1\right)$, $n_3 \ge 0$ OK. $\int_F f\,dF = \int_A f(r(x,y))\, r'_x \times r'_y\,dA =$

$= \int_{0\,(x)}^{1}\int_{-1\,(y)}^{1} (0, y\sqrt{1-y^2}, 1-y^2)\left(0, \frac{y}{\sqrt{1-y^2}}, 1\right)dy\,dx = \int_0^1\int_{-1}^{1} (y^2 + 1 - y^2)\,dy\,dx = |A| = 2$

5° a) $F$ az $x^2 + \frac{y^2}{4} \le 1$, $z = 1$ felület, $n = (0,0,1)$, $\operatorname{rot} f = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^2 & 2x & z^2 \end{vmatrix} = (0,0,2)$

$\oint_L f\,dr = \int_F \operatorname{rot} f\,dF = \int_F 2|dF| = 2|F| = 2\cdot\pi\cdot 1\cdot 2 = 4\pi$ (az $a, b$ féltengelű ellipszis területe $\pi ab$)

b) $f: \mathbb{R}^d \to \mathbb{R}^d$ feltenem diffható $a \in \mathbb{R}^d$ egy környezetében és $\det f'(a) \ne 0$. Akkor $\exists\, a \in U \subset \mathbb{R}^d$, $f(a) \in V \subset \mathbb{R}^d$ nyílt halmazok, hogy $f: U \to V$ bijekció, $f^{-1}$ diffható $f(a)$-ban és $(f^{-1})'(f(a)) = (f'(a))^{-1}$




6° a) $x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$ gömbfelület

b) $\int_F \operatorname{rot} f\,dF = \int_V \underbrace{\operatorname{div}(\operatorname{rot} f)}_{0}\,dV = 0$

Gauss-Ostrogradszkij