Controlled Asynchronism Improves Error Exponent

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Analízis Szeminárium, BME – 2015

Definition of Channel, Encoder, Decoder

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- The Channel is described by a Stochastic Matrix $W(y|x)$ $x \in \mathcal{X}$, $y \in \mathcal{Y}$.

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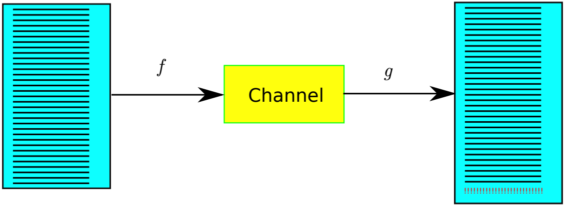

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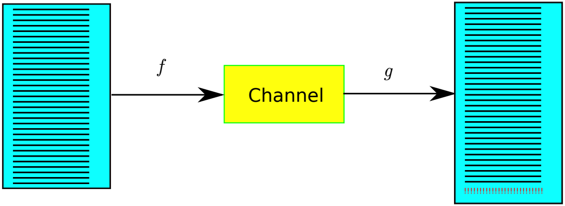

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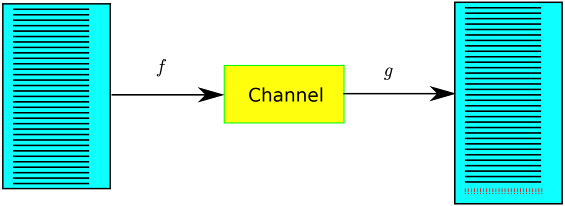
- There is a coding function f and a decoding function g
- The Codebook consists $M = 2^{nR}$ codeword. Where R is called as the rate of the code.

Definition of Error



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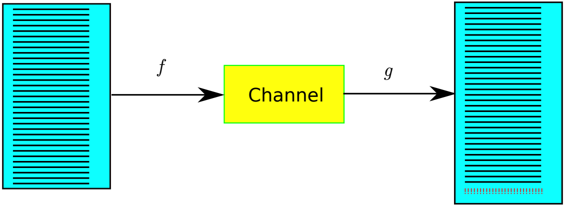
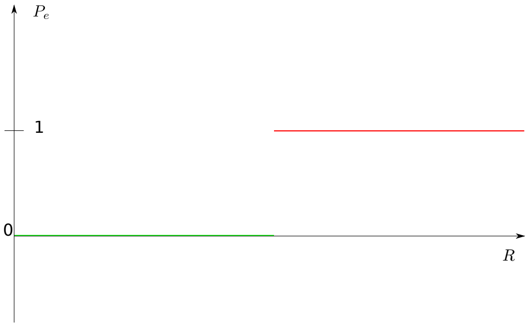


- It can be supposed, that the message are uniformly on $\{1, 2, \ldots, M\}$
- So, the *Error Probability* is

$$P_e = \frac{1}{M} \sum_{m=1}^{M} \Pr\{g(\mathbf{y}) \neq m | \mathbf{x} = f(m)\}$$

Definition Of Capacity, Error Exponent



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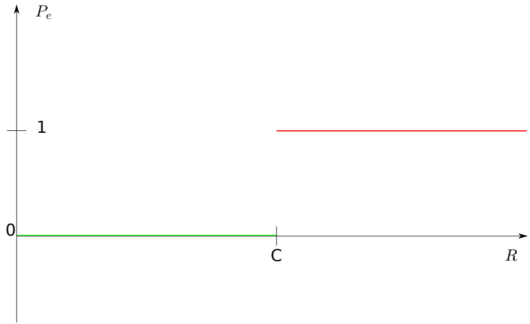

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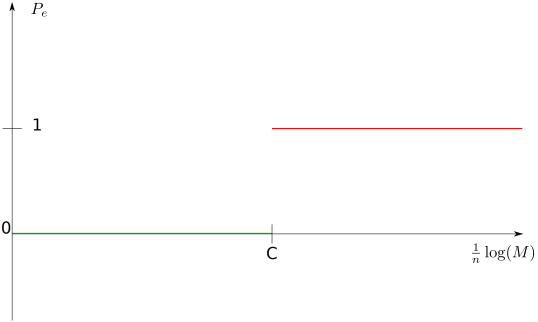

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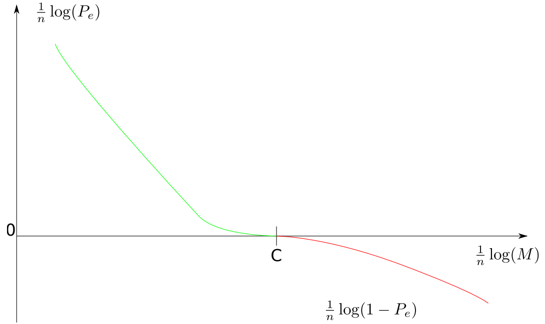

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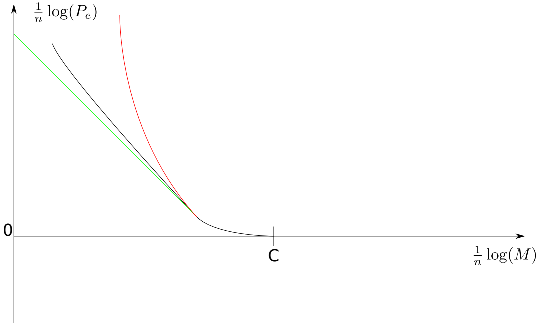

- Represent, the error probability as the function of Rate.
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- The logarithm of error is the *Error Exponent*
- The Error Exponent can be bounded.

Definition of Multiple Access Channel

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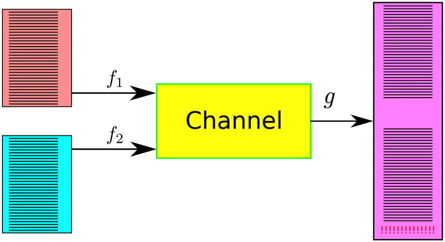

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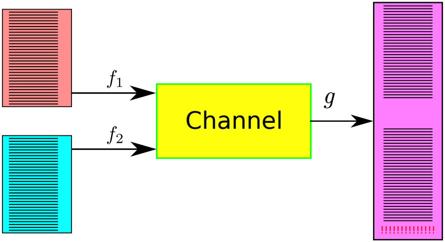

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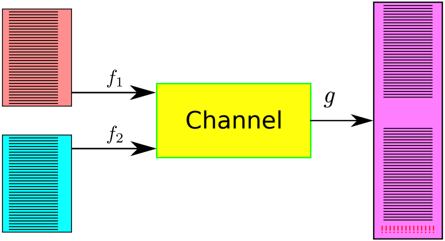
Definition of Error in Multiple Access System



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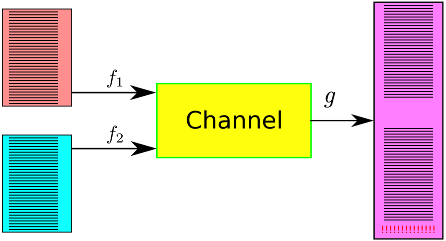
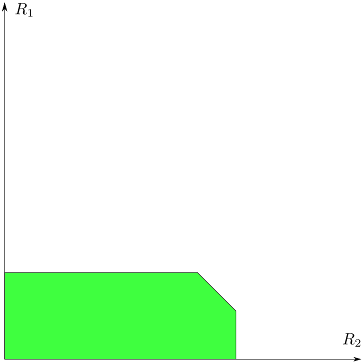


- The messages are uniform on $\{1,2,\ldots,M_1\}$, $\{1,2,\ldots,M_2\}$
- The Error Probability is

$$\frac{1}{M_1 M_2}\sum_{m_1=1}^{M_1}\sum_{m_2=1}^{M_2}\Pr\{g(\mathbf{z})\neq(m_1,m_2)|\mathbf{x}=f_1(m_1),\,\mathbf{y}=f_2(m_2)\}$$

Definition Of Capacity of the Multiple Access Channel



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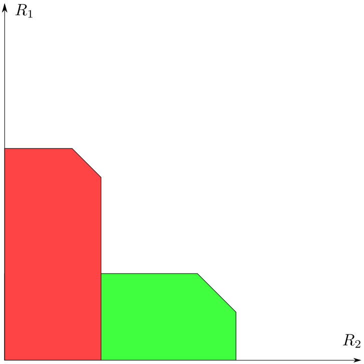

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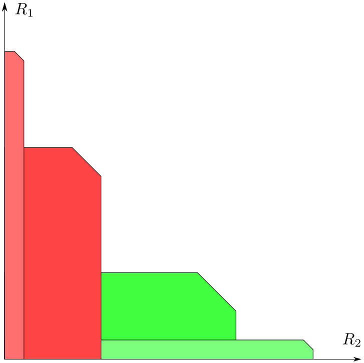

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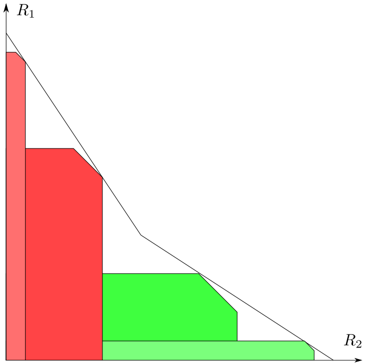

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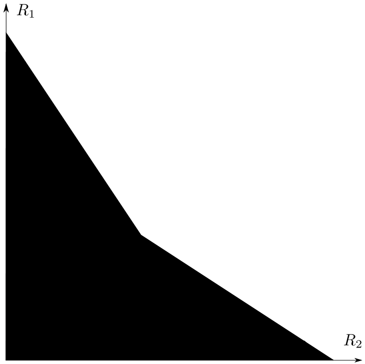

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- The union of these pentagos are not necessarily convex.

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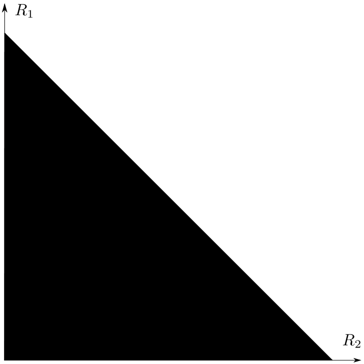

- If the input distributions are fixed, then a pentagon can be achieved.
- The union of these pentagos are not necessarily convex.
- In synchronous case, the convex closure can be taken.

Definition of Error Exponent, Previous work

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- For synchronous MAC. The first lower bound -called random coding bound- was given by Slepian and Wolf[10], later improved by Gallager[14], Dychakov[3], Pokorny and Wallmeier[11], Liu and Hughes[5], Nazari [8].

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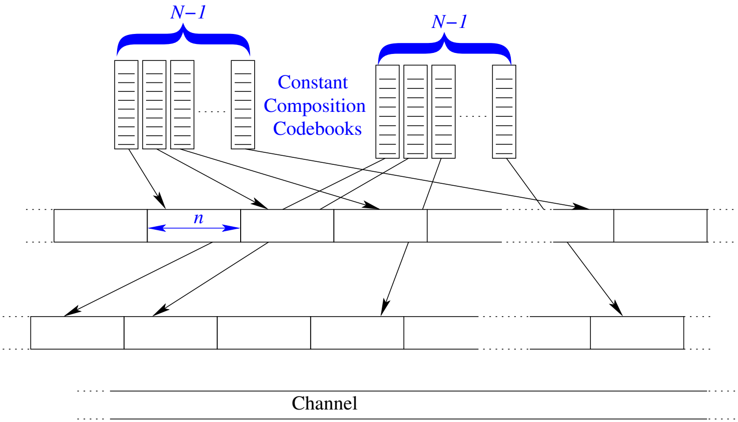
- Upper bounds were given by Haroutunian[2], Nazari et Al. [7].

The Model



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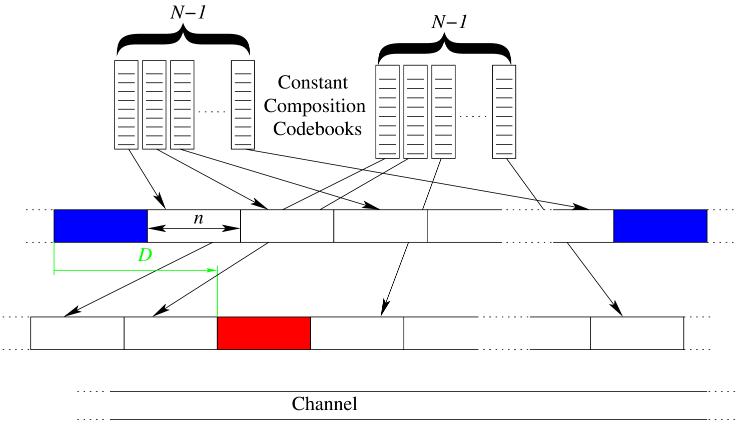

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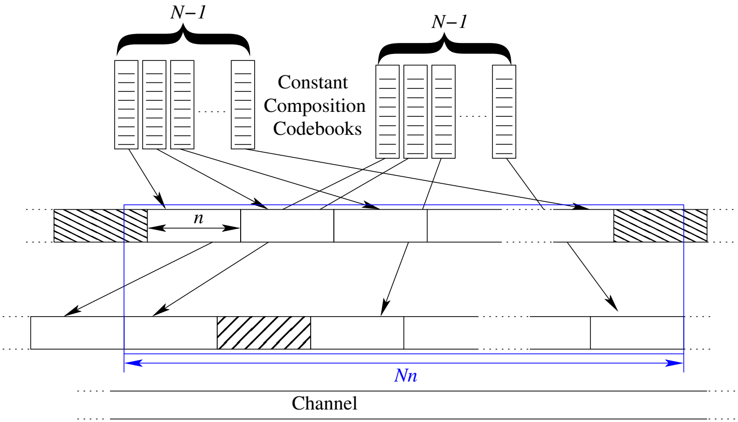

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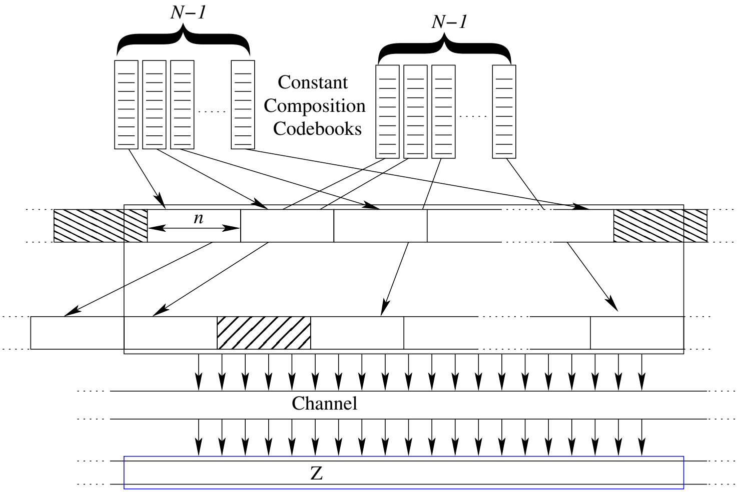
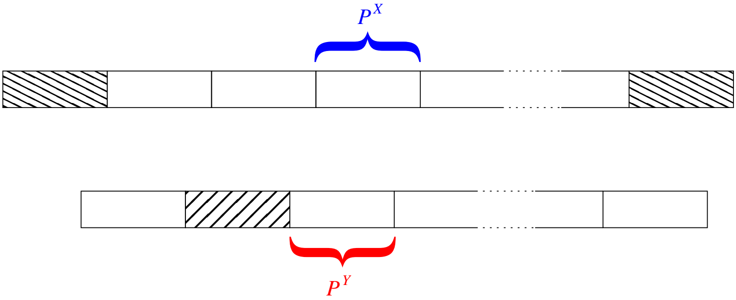


Definition of Subtypes



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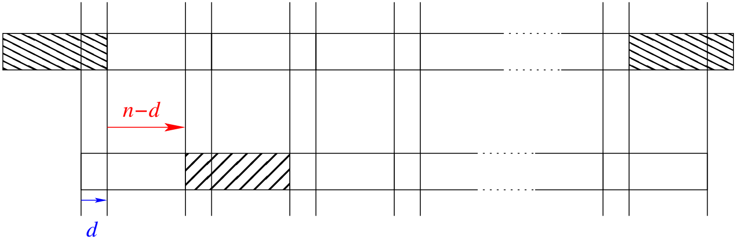

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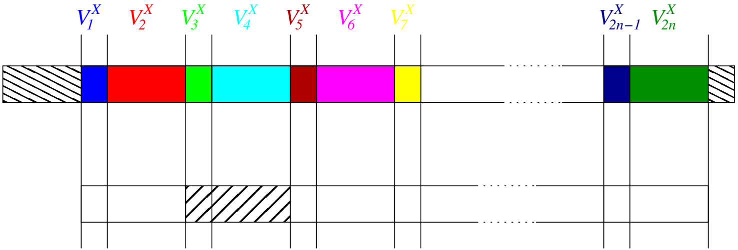

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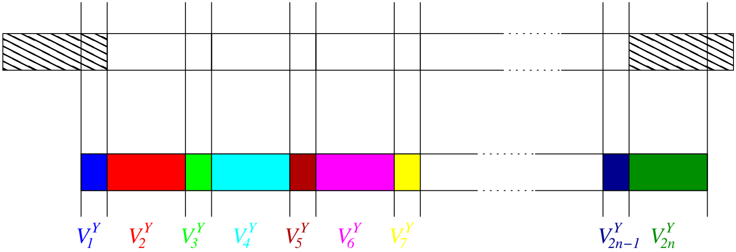

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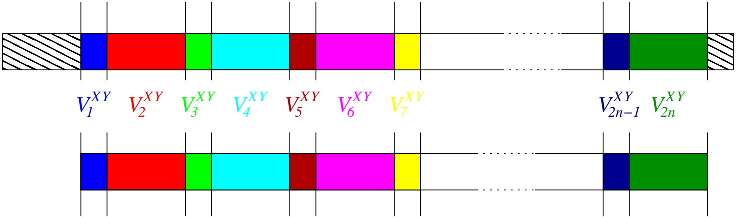
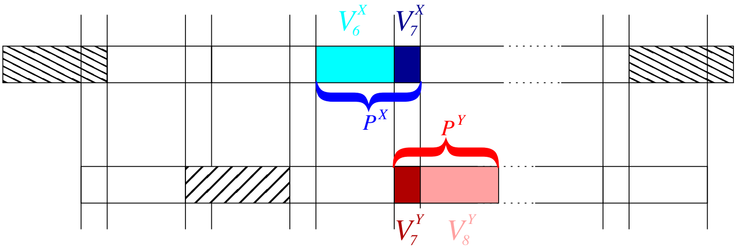


The Z-property of the Subtypes



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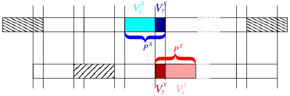

- Merging V_6^X and V_7^X gives P^X (similarly merging V_5^Y and V_6^Y gives P^Y): $(n-d) \cdot V_6^X(a) + d \cdot V_7^X(a) = nP^X(a) \quad \forall a \in \mathcal{X}$

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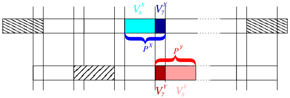

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- The probability of the event that an $\mathbf{x}$ uniformly chosen from $\mathcal{T}^n_{P^X}$ has type V_6^X in the first $n-d$ symbols and has type V_7^X on the last d symbols is

$$\frac{\left|\mathcal{T}^n_{V_6^X}\right|\cdot\left|\mathcal{T}^n_{V_7^X}\right|}{\left|\mathcal{T}^n_{P^X}\right|} \approx \frac{2^{(n-d)\,\mathrm{H}(V_6^X)}2^{d\,\mathrm{H}(V_7^X)}}{2^{n\,\mathrm{H}(P^X)}} = 2^{-nF(V_6^X,V_7^X)}$$

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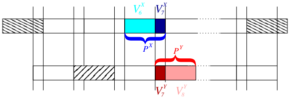

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- The non-negative exponent $F(V_6^X, V_7^X)$ is called Jensen-Shannon divergence.

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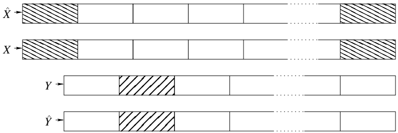
  the measure of their dependence.
- The existence of such codebook is proved by random choosing.

Packing Lemma in Synchronous Multi User Case

- In synchronous multi-user case the maximal multi Information decoder works if the logarithm of number of quadruples $(\mathbf{x},\mathbf{y}),(\hat{\mathbf{x}},\hat{\mathbf{y}})$ with type V is also inversely proportional to

$$\mathrm{I}_V(X \wedge \hat{X} \wedge Y \wedge \hat{Y})) =$$
$$= \mathrm{H}_V(X) + \mathrm{H}_V(\hat{X}) + \mathrm{H}_V(Y) + \mathrm{H}_V(\hat{Y}) - H_V(X,\hat{X},Y,\hat{Y}) =$$
$$= \mathrm{D}(V_X \cdot V_{\hat{X}} \cdot V_Y \cdot V_{\hat{Y}} \| V_{X\hat{X}Y\hat{Y}})$$

Packing Lemma for asynchronous Codebooks



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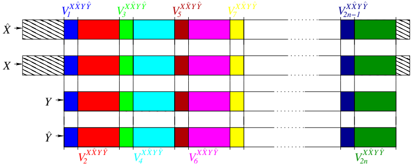

- We need a constant composition library, that for each possible delay d, the negative logarithm of number of possible quadruples of codeword sequences with any subtype sequence $V_1, V_2, \ldots, V_{2N}$ is approximately equal to

$$d\,\mathrm{I}_{V_1}(X;\hat{X};Y;\hat{Y}) + (n-d)\,\mathrm{I}_{V_2}(X;\hat{X};Y;\hat{Y}) + \ldots$$
$$+\,(n-d)\,\mathrm{I}_{V_{2N}}(X;\hat{X};Y;\hat{Y}) +$$
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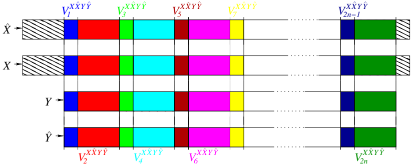

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$$+ (n-d)\,\mathrm{I}_{V_{2N}}(X;\hat{X};Y;\hat{Y}) +$$
$$+ \sum_{k=1}^{N} \left[F(V_{2k}^X, V_{2k+1}^X) + F(V_{2k-1}^Y, V_{2k}^Y)\right]$$

- The existence is proved by random selection (Packing Lemma).

Maximal Multi Information Decoder

One–One

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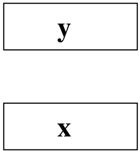

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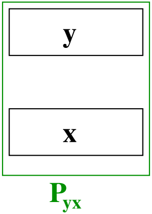

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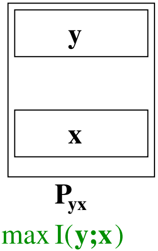
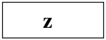


Maximal Multi Information Decoder

Sychronous



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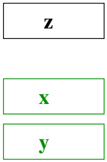

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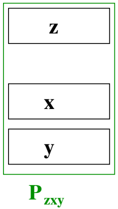

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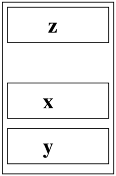

$\mathbf{P}_{\mathbf{zxy}}$

max I(**z;x;y**)

$$I(\mathbf{z};\mathbf{x};\mathbf{y}) = \mathrm{H}(\mathbf{z}) + \mathrm{H}(\mathbf{x}) + \mathrm{H}(\mathbf{y}) - \mathrm{H}(\mathbf{z},\mathbf{x},\mathbf{y})$$

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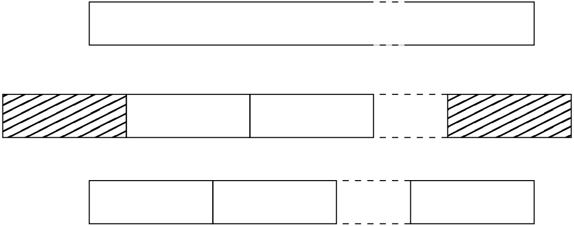

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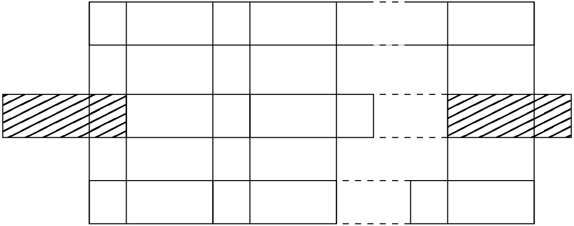

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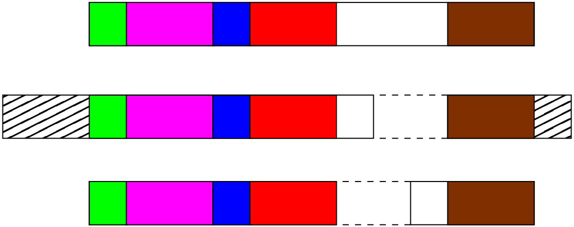

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Maximal Multi Information Decoder

$$\max\left(d\,\mathrm{I}(\mathbf{z};\mathbf{x};\mathbf{y}) + (n-d)\,\mathrm{I}(\mathbf{z};\mathbf{x};\mathbf{y}) + d\,\mathrm{I}(\mathbf{z};\mathbf{x};\mathbf{y}) + (n-d)\,\mathrm{I}(\mathbf{z};\mathbf{x};\mathbf{y}) + d\,\mathrm{I}(\mathbf{z};\mathbf{x};\mathbf{y}) \right)$$

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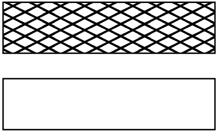

Error exponent of the Synchronous Channels

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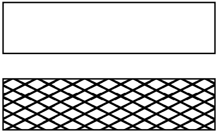
$$E_2(R_1, R_2) = \min_{V_{X_1 X_2 Y U}} \mathrm{D}(V_{Y X_1 X_2|U} \| W \times P_{X_1 X_2|U} | P_U) + |\, \mathrm{I}_V(X_2 \wedge Y, X_1) - R_2|^+$$



Error exponent of the Synchronous Channels

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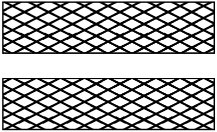
$$
\begin{aligned}
E_{12}(R_1, R_2) = \min_{V_{X_1 X_2 Y U}} & \mathrm{D}(V_{Y X_1 X_2|U} \| W \times P_{X_1 X_2|U} | P_U) \\
& + \left| \mathrm{I}_V(X_1 \wedge X_2 \wedge Y) - R_1 - R_2 \right|^+
\end{aligned}
$$



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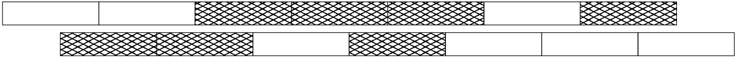
Definition of the Error Pattern



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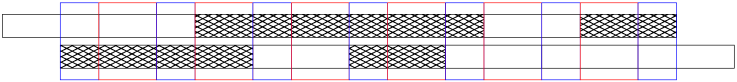

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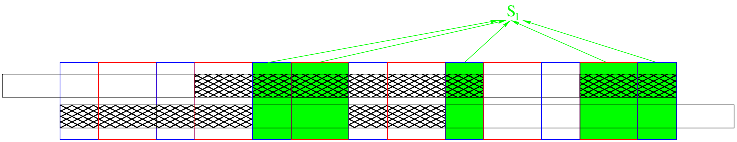

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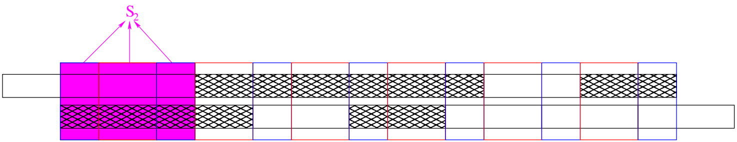

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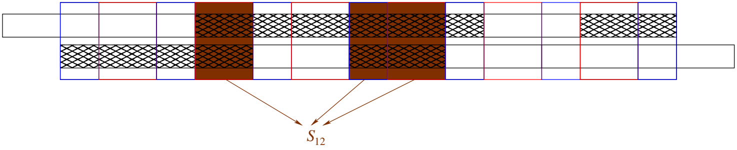

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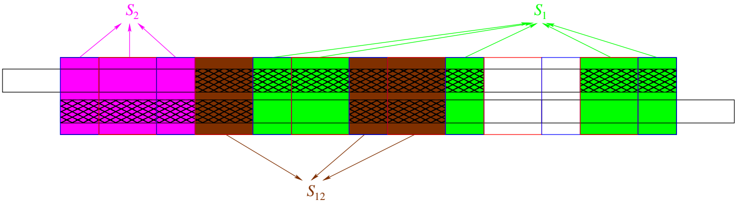

$$\mathbf{S} = (S_1, S_2, S_{12})$$

- The error probability is union of the all possible not empty error pattern. So,

$$P_e^d \leq \sum_{\mathbf{S}} P_e^d(\mathbf{S})$$

The Intermediate Error Exponent

Theorem (Farkas, Kói)

For an Error Pattern with subblocks $\mathbf{S} = (S_1, S_2, S_{12})$ *and delay* d *the probability of error is:* $P_e^d(\mathbf{S}) \leq 2^{-nE_{\mathbf{S}}^d(R_1,R_2)}$ *where*

$$nE_{\mathbf{S}}^d(R_1,R_2) = \min_{\mathbf{V}} \sum_{k\in[2N]} n_k \, \mathrm{D}(V_k \| P^{XYZ}) + \\ + \left| \sum_{k\in S_1} n_k (\mathrm{I}_{V_k}(X;YZ) - R_1) + \right. \\ + \sum_{k\in S_2} n_k (\mathrm{I}_{V_k}(Y;XZ) - R_2) + \\ \left. + \sum_{k\in S_{12}} n_k (\mathrm{I}_{V_k}(X;Y;Z) - R_1 - R_2) \right|^+$$

Problems With The Previous Exponent

The previous exponent cannot be numerically computed because:

- Though N can be constant cannot be too small, while we do not send messages in synchronization blocks.

$$
\begin{aligned}
nE^d_{\mathbf{S}}(R_1, R_2) = \min_{\mathbf{V}} \sum_{k\in[2N]} n_k \, \mathrm{D}(V_k \| P^{XYZ}) + \\
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+ \sum_{k\in S_2} n_k(\mathrm{I}_{V_k}(Y;XZ) - R_2) + \\
+ \sum_{k\in S_{12}} n_k(\mathrm{I}_{V_k}(X;Y;Z) - R_1 - R_2) \Bigg|^{+}
\end{aligned}
$$

Problems With The Previous Exponent

The previous exponent cannot be numerically computed because:

- The number possible error pattern exponential in N, so there are lots of choices for S_1, S_2, S_{12}

$$nE^d_{\mathbf{S}}(R_1, R_2) = \min_{\mathbf{V}} \sum_{k\in[2N]} n_k\,\mathrm{D}(V_k \| P^{XYZ}) + \\ + \left| \sum_{k\in S_1} n_k(\mathrm{I}_{V_k}(X;YZ) - R_1) + \right. \\ + \sum_{k\in S_2} n_k(\mathrm{I}_{V_k}(Y;XZ) - R_2) + \\ \left. + \sum_{k\in S_{12}} n_k(\mathrm{I}_{V_k}(X;Y;Z) - R_1 - R_2) \right|^+$$

Problems With The Previous Exponent

The previous exponent cannot be numerically computed because:

- The length of subtype sequence $V_1, V_2, \ldots, V_{2N}$ -with the disturbing Z-property- also grows with N.

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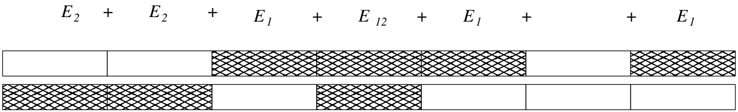
Monotonicity in the Synchronous system

- At synchronous system if you minimalize the exponent on the error pattern.



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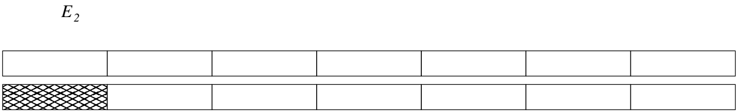

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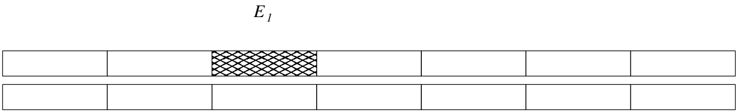

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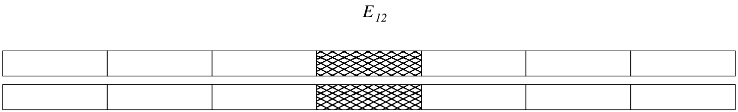

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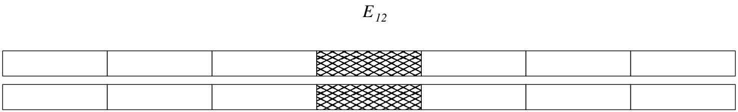

- So the minimal exponent you get:

$$E_s = \min[E_1, E_2, E_{12}]$$

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- As we have seen, in synchronous system the minimal error pattern length is always 1 block.

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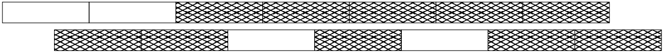

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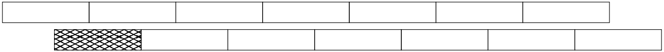

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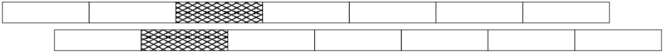

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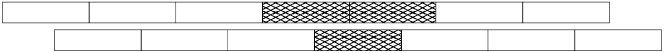

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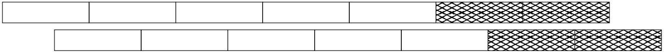

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Definition

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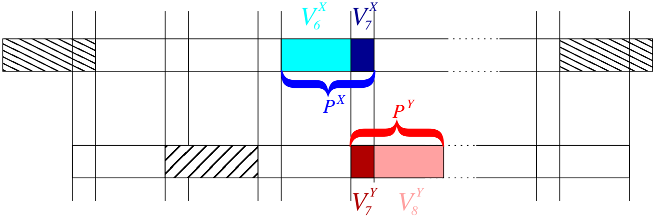
- For sufficiently large n, the number of δ-balanced sequences has the same exponent as the size of the type-class
- There exist codebooks with the needed properties chosen from δ-balanced codewords.

Eliminating the Z-property, Convexity

- With the Pinsker inequality and continuity of the exponent we only need to minimize over distributions which marginals are P, Q. So, the Z-property is eliminated.



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$$+ \sum_{k\in S_{12}} n_k \, \mathrm{D}(V_k \| P^{XYZ}) + \Bigg| \sum_{k\in S_1} n_k (\mathrm{I}_{V_k}(X;YZ) - R_1) +$$
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$$\min_{V_1,V_2,V_{12}\in\mathcal{V}^P} l_1\,\mathrm{D}(V_1\|P^{XYZ}) + l_2\,\mathrm{D}(V_2\|P^{XYZ})$$

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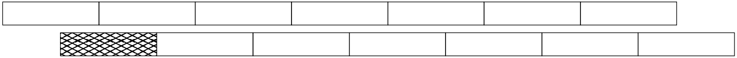
Applying the Weak Monotonicity

- Recall the weak monotonicity. That means l_1 and l_2 can have the length of at most two subblock, depending on the start index i_0 and length K of the error pattern.



$$l_1 = 0, l_2 = n, l_{12} = 0$$

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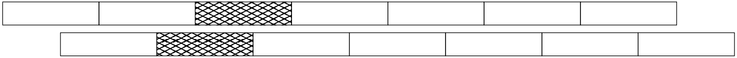

$$l_1 = d, l_2 = d, l_{12} = n - d$$

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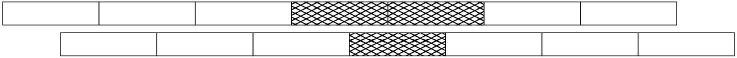

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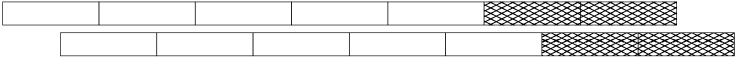

$$l_1 = n - d, l_2 = n - d, l_{12} = n + d$$

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(a) Odd K, odd i_0: $l_1 = l_2 = d$, $l_{12} = n - d + \frac{K-3}{2}n$

(b) Odd K, even i_0: $l_1 = l_2 = n - d$, $l_{12} = d + \frac{K-3}{2}n$

(c) Even K, odd i_0: $l_1 = 0$, $l_2 = n$,$l_{12} = \frac{K-2}{2}n$

(d) Even K, even i_0: $l_1 = n$, $l_2 = 0$,$l_{12} = \frac{K-2}{2}n$

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$$+l_{12}\,\mathrm{D}(V_{12}\|P^{XYZ}) + \Big|l_1(\mathrm{I}_{V_1}(X;YZ) - R_1)+$$
$$+l_2(\mathrm{I}_{V_2}(Y;XZ) - R_2) + l_{12}(\mathrm{I}_{V_{12}}(X;Y;Z) - R_1 - R_2)\Big|^+,$$

If $d = \frac{n}{2}$, $R_1 = R_2 = R$ and the channel is symmetric. With $\frac{K-2}{2} = L$.

Applying the Weak Monotonicity

- Recall the weak monotonicity. That means l_1 and l_2 can have the length of at most two subblock, depending on the start index i_0 and length K of the error pattern.

$$\min_{V_1, V_{12} \in \mathcal{V}^P} n\,\mathrm{D}(V_1 \| P^{XYZ}) + \\ + Ln\,\mathrm{D}(V_{12} \| P^{XYZ}) + \Big| n(\mathrm{I}_{V_1}(X;YZ) - R) + \\ + Ln(\mathrm{I}_{V_{12}}(X;Y;Z) - 2R) \Big|^+,$$

If $d = \frac{n}{2}$, $R_1 = R_2 = R$ and the channel is symmetric. With $\frac{K-2}{2} = L$.

Controlled Asynchronous system

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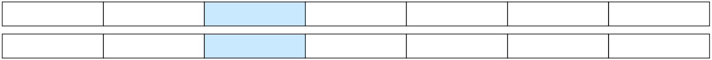

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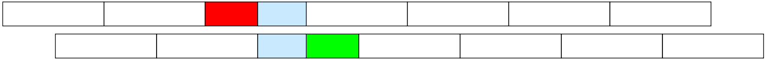

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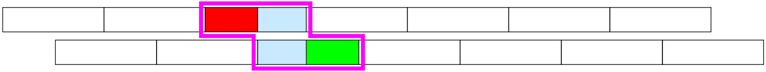

$$P_e^{as} = \max_d P_e^d, \qquad P_e^{cas} = \min_d P_e^d$$

Binary Adder Channel

- The synchronous exponent of Liu and Hughes and this new asynchronous exponent was compared on the *binary adder channel*[(a),(b)], with $R_1 = R_2^{(b)}$, $d = n/2^{(b)}$ and $N = 20$.

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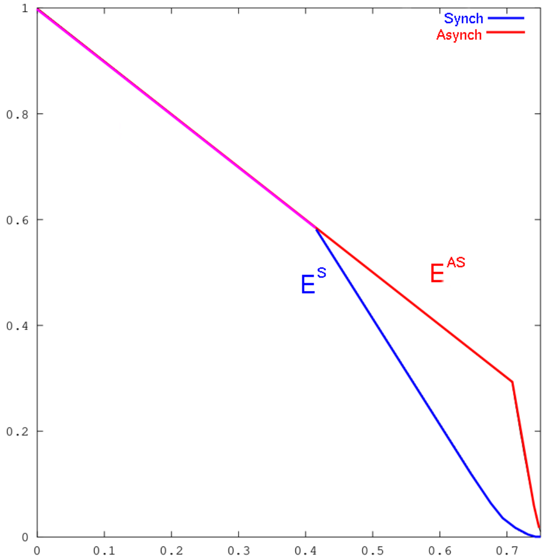
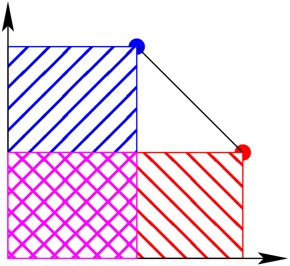
  (a) The channel is deterministic, hence only joint distribution with $V_{Z|XY} = W$ have finite divergence.$\Rightarrow \mathrm{D}(V_{Z|XY} \| W) = 0$
  (b) The resulting exponent is much more simpler.

Result on Binary Adder Channel



An alternative to splitting

- For rate pairs dominated by either corners of the pentagon successive cancellation is possible.



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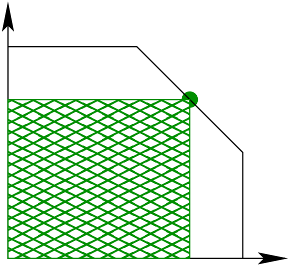

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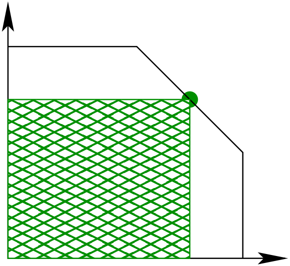

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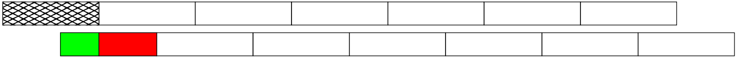

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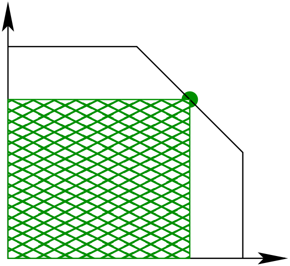

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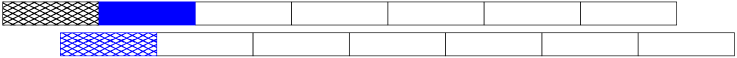

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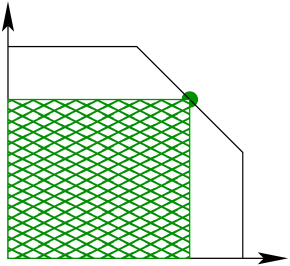

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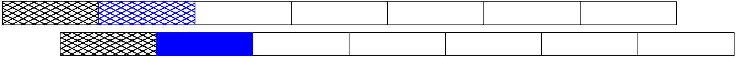

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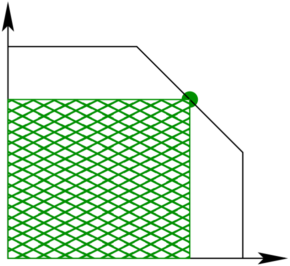

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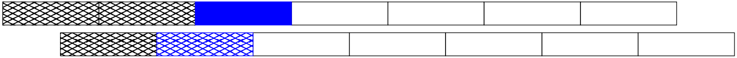

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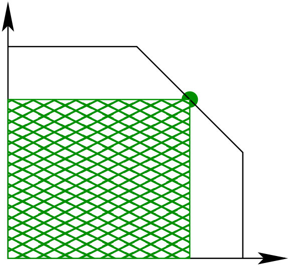

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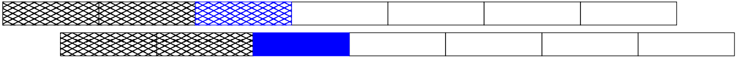

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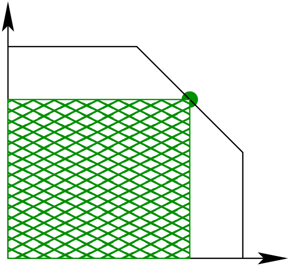

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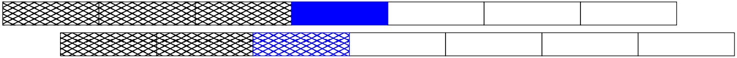

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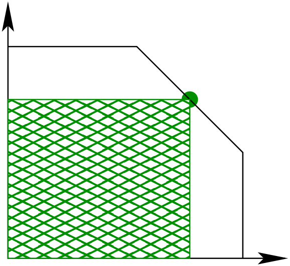

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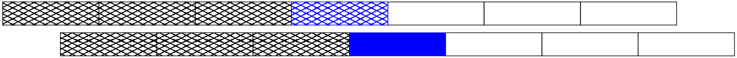

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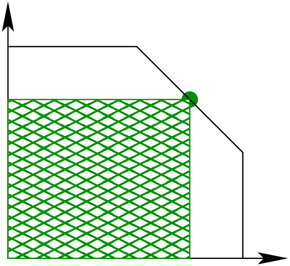

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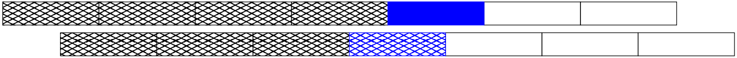

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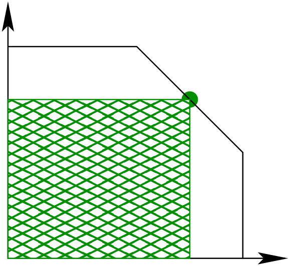

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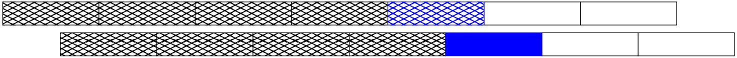

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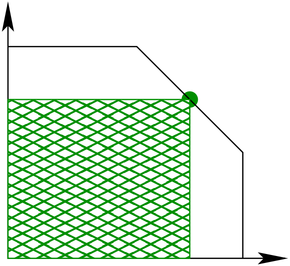

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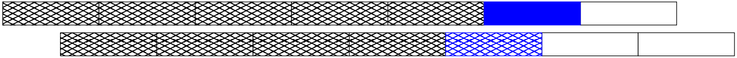

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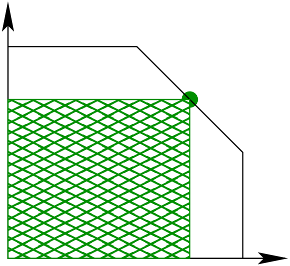

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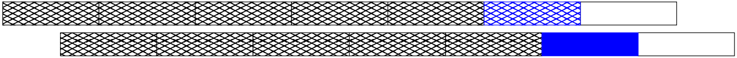

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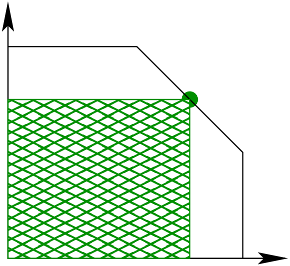

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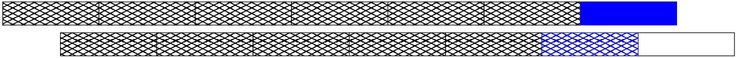

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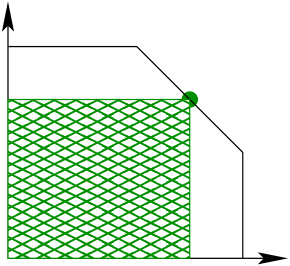

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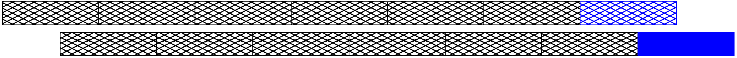

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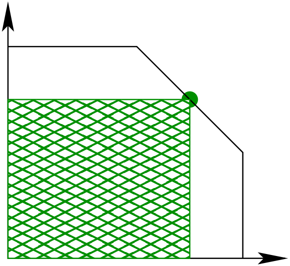

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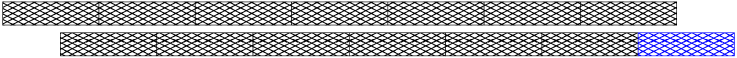

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Thank You For Your Attention

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